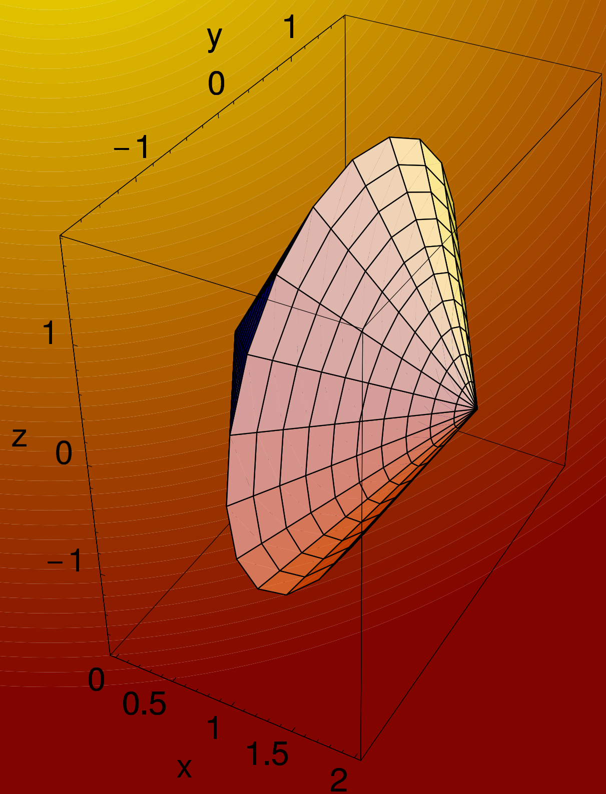
P101 Mathematics for Physics



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Mathematics for Physicists
DRAFT

Niels Walet, Fall 2002

Last changed on October 30, 2002

Contents

Introduction

Here you find the lecture notes for the first semester of the course "Mathematics for Physicsists". These notes are terse, but should cover more-or-less what has been said in class. You can use them as a guide to the material you are expected to be able to deal with, and we give ample reference to the two textbooks (Lambourne and Tinker, "Basic Mathematics for the Physical Sciences", denoted as 1.xxx, and Tinker and Lambourne, "Further Mathematics for the Physical Sciences", denoted as 2.xxxx). You'll notice that we jump through those books in a rather random order, but your are *expected* to read up on those parts that you find difficult, or are not covered in enough detail in the notes.

Niels Walet, Manchester, 2002

Chapter 1

Introduction

1.1 Why mathematics for physics?

At first you may ask yourself the question why combine mathematica and physics, if they can be taught as almost fully independent subjects in your A-level courses.

The answer is of course "because they are taught as independent subjects"! Much of mathematics – most of the calculus and algebra discussed in this course – was originally developped to deal with the problems arising from the development of physics in the 18th and 19th century. Actually, it was often hard to distinguish a mathematician from a physicist!

1.2 Mathematics as the language for physics

That brings us automatically to our next subject, the fact that part of mathematics was developped to describe real-world problems, and thus is the natural language of physics. Let us study this issue by looking at a number of examples.

Example 1.1:

Describe the motion of a particle under a constant force

Solution:

Example 1.2:

Discuss the equilibrium of forces in a spiders web.

Solution:

Example 1.3:

Solution:

Chapter 2

Revision

2.1 Powers, logs, exponentials

L&T, 1..6

2.1.1 Powers

L&T, 1..1.2.4

Here we summarise the properties of the powers.

2.1.2 The product of two powers

First of all the product of two powers,

$$a^x a^y = a^{x+y} \quad , \tag{2.1}$$

e.g., $3^2 3^6 = 3^8$, and $3^{1/2} 3^{3/2} = 3^2$ (we see that x and y do *not* have to be integers (whole numbers)). Question: Evaluate $5^{7/10} 5^{3/10}$.

The power of a power

If we take the power of a power, we multiply the exponents,

$$(a^x)^y = a^{xy} \tag{2.2}$$

e.g., $(2^3)^2 = 8^2 = 64 = 2^{3\times 2} = 2^6 = 64$. This again works for x, y not integers. Question: Evaluate $2^{1/4} 4^{3/8}$.

Relation with roots

If the exponent is $1/n$ we are taking the nth root of a,

$$a^{1/n} = \sqrt[n]{a}, \tag{2.3}$$

e.g., $2^{\frac{1}{2}} = \sqrt{2}$, $2^{\frac{1}{3}} = \sqrt[3]{2}$. If $x = a^{\frac{1}{n}}$ then $x^n = a$. This can be shown by taking both sides to the power n,

$$x^n = \left(a^{1/n}\right)^n = a^1 = a.$$

The number n is often taken to be an integer, but it does not have to be. (E.g., $(3^{1/9.5})^{9.5} = 3$.)

Zeroth power of a number

If we take a number to the power zero, we find

$$a^0 = 1 \text{ for any } a > 0. \tag{2.4}$$

This follows from $a^0 a^x = a^{x+0} = a^x$, and therefore $a^0 = 1$. (Note that there is a slight problem with 0^0: $0^x = 0$ for $x > 0$. One *usually* defines $0^0 = 1$.)

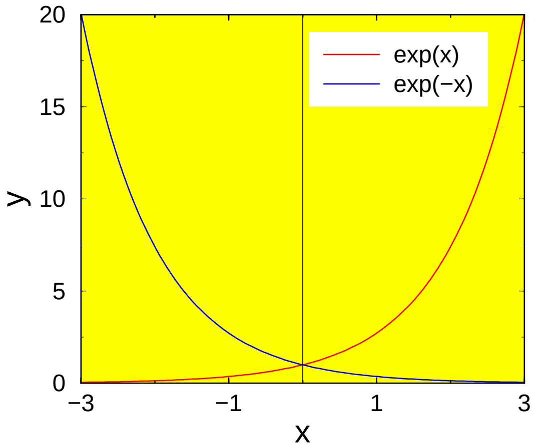

Figure 2.1: A plot of the exponential $\exp(x)$ and $\exp(-x)$.

Negative powers and fractions

If we take a number to a negative power, we write the result as a fraction involving a positive power,

$$a^{-x} = \frac{1}{a^x} \tag{2.5}$$

since $a^{-x}a^x = a^{-x+x} = a^0 = 1$. Therefore $a^{-x} = \frac{1}{a^x}$. E.g., $2^{-1} = 1/2$.

Common error

Remember that

$$a^{x+y} = a^x a^y \quad \text{CORRECT!!!}, \tag{2.6}$$

and not

$$a^{x+y} \neq a^x + a^y \quad \text{WRONG!!!} \tag{2.7}$$

As an example, $2^{3+5} = 2^8 = 64$, but $2^3 + 2^5 = 8 + 32 = 40$.

2.1.3 Exponential Function

L&T, 1..6.2

The exponential function is a special case of a power, where $y = e^x$, with $e = 2.71828....$ (Euler's number). One also writes $\exp(x)$ instead of e^x.

As we can see from Fig. 2.1, e^x is never less than 0 for any x. From the properties of powers we know that $e^{-x} = \frac{1}{e^x}$. This function is also shown in Fig. 2.1, and is positive as well.

Differential (derivative w.r.t. x) of e^x is e^x, i.e.,

$$\boxed{\frac{de^x}{dx} = e^x \quad .}$$

(This is the only function with the property that the derivative equals the function itself.)

If $y = e^{ax}$ then $\frac{dy}{dx} = ae^{ax}$ (this is a form of the chain rule, which will be discussed later), e.g., if $y = 3e^{7x}$ then $\frac{dy}{dx} = 3 \times 7e^{7x} = 21e^{7x}$.

Example 2.1:

Discuss exponential growth/decay.

Solution:

Exponential growth or decay is ruled by the form $N(t) = N_0 e^{at}$. For $a < 0$ we have decay, for $a > 0$ we have growth. From the derivative, $\dfrac{dN}{dt}(t) = N_0 a e^{at} = aN(t)$ we see that this arrises when the change in N is proportional to the number present. Examples are population growth, radioactive deay, ...

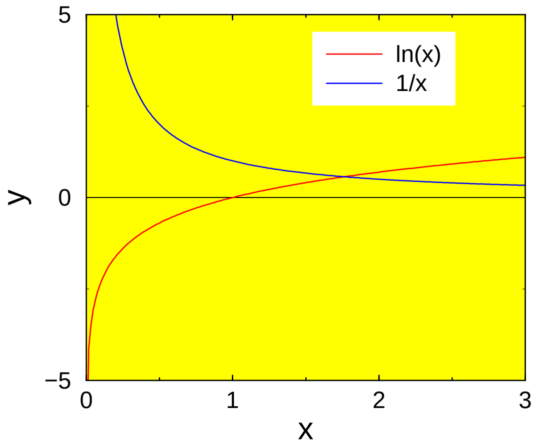

Figure 2.2: A plot of the natural logarithm $\ln(x)$ and its derivative $1/x$.

2.1.4 The Logarithmic Function

L&T, 1..6.3

Relation between Logs and Exponentials

L&T, 1..6.3.1

The inverse f^{-1} of a function f is defined such that if $y = f(x)$, then $x = f^{-1}(y)$.

The functions $\ln(x)$ and $\exp(x)$ are the inverse functions of each other. This means that if $y = \ln(x)$ then $x = e^y$. The reverse is also true, if $x = e^y$ then $y = \ln x$. Clearly it follows that, using these relations,

$$\begin{aligned} \exp(\ln x) &= e^{\ln x} = e^y = x, \\ \ln(\exp y) &= \ln(e^y) = \ln x = y. \end{aligned}$$

A graph of the logarithm is shown in Fig. 2.2. If we swap the x and y axes, we recognise the exponential. Normally we use logs to base e (inverse of e^x)- called natural logarithms, hence the name $\ln(x)$, but we also write

$$\log(x) = \ln(x) \quad .$$

Logs to other bases

L&T, 1..6.3.2

Just as $y = \ln x \Rightarrow x = e^y$ for the logarithm corresponding to base e (i.e., the inverse of e^x) for other bases we have $y = \log_a x \Rightarrow x = a^y$. Here we use the notation that if we mean log to base, say, 10 we write $\log_{10}(x)$, i.e., if $y = \log_{10}(x)$, $x = 10^y$.

It may help you to remember that a logarithm tries to extract a power from a number, e.g. the $\log_1 0$ extract the power of 10 from a number.

Change of base

Using this we can change from one base to another. Let $y = \log_{10} x$, then $x = 10^y$. Now let $b = \ln 10 (= \log 10)$, so $10 = e^b$. Therefore $x = (e^b)^y = e^{by}$, so $by = \ln x$, $y = \frac{\ln x}{b} = \frac{\ln x}{\ln 10}$. Hence $\log_{10} x = \frac{\ln x}{\ln 10}$. Question: Determine α such that $\log_{10}(x) = \alpha \log_2(x)$.

Differential of $\ln x$

If $y = \ln x$ then

$$\frac{dy}{dx} = \frac{1}{x} \quad .$$

(Remember that the differential of $\ln x$ is $1/x$, not the integral! This is a common error!)

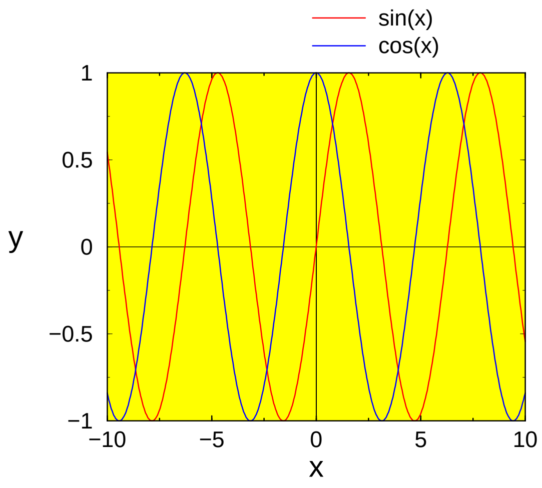

Figure 2.3: A plot of the sine and cosine.

Log of a product

Using the fact that $e^{x_1}e^{x_2} = e^{x_1+x_2}$, i.e., the product of exponents is the exponent of the sum, we conclude that the inverse relation holds for logarithms. Thus, the logarithm of a product is the sum of the logarithms,

$$\ln(y_1y_2) = \ln(e^{x_1}e^{x_2}) = \ln e^{x_1+x_2} = x_1 + x_2.$$

Example 2.2:

The magnitude of a start is defined as $m = \log_{10}(I/I_0)$. Explain how I changes if m increase by one unit.

Solution:

The new intensity satisfies $\log_{10}(I^{\text{new}}/I_0) = \log_{10}(I^{\text{old}}/I_0) + 1$. Using the properties of the logarithms, we find that

$$\begin{aligned}
\log_{10}(I^{\text{new}}/I_0) &= \log_{10}(I^{\text{old}}/I_0) + \log_{10} 10 \\
\log_{10}(I^{\text{new}}/I_0) &= \log_{10}(10I^{\text{old}}/I_0) \\
(I^{\text{new}}/I_0) &= (10I^{\text{old}}/I_0) \\
I^{\text{new}} &= 10I^{\text{old}}
\end{aligned}$$

Example 2.3:

An unresolved doube-star has magnitude 7. Find the individual magnitudes, assuming that both stars have the same one.

Solution:

Since intensities add up, we have $7 = \log_{10}(2I/I_0) = \log_{10}(2) + \log_1 0(I/I_0) = \log_{10}(2) + m$. Thus we conclude that $m = 7 - \log_{10}(2) = 6.69897$.

2.2 Trigonometric functions

L&T, 1.5.3.1

Trigonometric functions are the sine ($\sin(x)$), cosine ($\cos(x)$), tangent ($\tan(x) = \sin(x)/\cos(x)$), cotangent ($\cot(x) = 1/\tan(x)$), secans ($\sec(x) = 1/\cos(x)$) and cosecans ($\mathrm{cosec}(x) = 1/\sin(x)$).

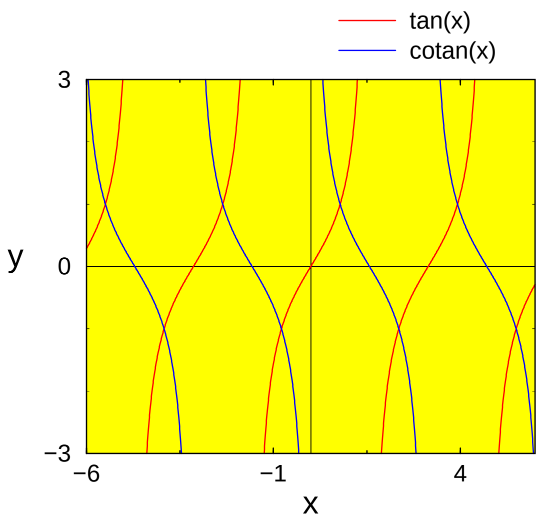

Figure 2.4: A plot of the tangent and cotangent.

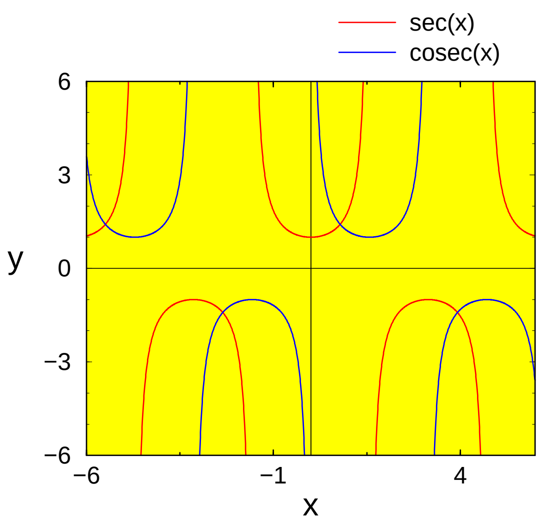

Figure 2.5: A plot of the secans and cosecans.

2.2.1 Trigonometric identities

L&T, 1.5.3.2

We shall assume that you are familiar with the sine and cosine of the sum of two angles,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B, \tag{2.8}$$
$$\cos(A+B) = \cos A\cos B - \sin A\cos B. \tag{2.9}$$

We also expect you to know that

$$\cos^2\theta + \sin^2\theta = 1 \tag{2.10}$$

for all θ. Substitute $A = B$ in Eq. (2.8), and find $\sin 2A = \sin A\cos A + \cos A\sin A$, and thus

$$\sin 2A = 2\sin A\cos A. \tag{2.11}$$

In (2.9) put $B = A$, $\cos 2A = \cos A\cos A - \sin A\sin A$, so

$$\cos 2A = \cos^2 A - \sin^2 A. \tag{2.12a}$$

However from (2.10) we have $\sin^2 A = 1 - \cos^2 A$ so we can rewrite (2.12a) as

$$\cos 2A = \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1. \tag{2.12b}$$

Similarly (left as exercise)

$$\cos 2A = 1 - 2\sin^2 A. \tag{2.12c}$$

Example 2.4:

Evaluate $\cos(75°)$.

Solution:

$$\begin{aligned}\cos(75°) &= \cos(45°+30°) = \cos 45°\cos 30° - \sin 45°\sin 30° \\ &= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \boxed{0.2588}\quad .\end{aligned}$$

Note: We shall use radians more often than degrees, $180° = \pi$ radians, so

$$\boxed{A° = \frac{A\times\pi}{180}\text{ radians.}}$$

E.g., $\cos 45° = \cos\frac{\pi}{4}$, $\sin 30° = \sin\frac{\pi}{6}$. Usually, if there is no degree sign (°) then the angle is specified in radians.

Example 2.5:

Show from the equations above that

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

Hint:

$\tan 2A = \frac{\sin 2A}{\cos 2A}$.

Other formulae

You will sometimes need other formulae such as

$$\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$$

(there are four of these), and

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

(there are three of these).

One formula you may not have seen before is

$$a\sin x + b\cos x = R\sin(x+\phi).$$

To find R and ϕ we use formula (2.8)and find

$$a\sin x + b\cos x = R[\sin x\cos\phi + \cos x\sin\phi] = R\cos\phi\sin x + R\sin\phi\cos x \quad .$$

We equate the coefficient of $\sin x$ and $\cos x$ on both sides of the equation, and find

$$a = R\cos\phi, \quad b = R\sin\phi \quad .$$

Therefore

$$a^2 + b^2 = R^2\cos^2\phi R^2 + \sin^2\phi = R^2,$$

and thus

$$\boxed{R = \sqrt{a^2+b^2} \quad .}$$

We also find

$$b/a = \frac{R\sin\phi}{R\cos\phi} = \tan\phi,$$

so

$$\tan\phi = \frac{b}{a}$$

and

$$\boxed{\phi = \tan^{-1}(b/a).}$$

($\tan^{-1}$ will be discussed later.)

Example 2.6:

Express $3\sin x + 2\cos x$ in the form $R\sin(x+\phi)$.

Solution:

We find $R\cos\phi = 3$, $R\sin\phi = 2$, $R^2\cos^2\phi = 9$, $R^2\sin^2\phi = 9$, $R^2(\cos^2\phi + \sin^2\phi) = 9 + 4 = 13$. Therefore $R^2 = 13$, $R = \sqrt{13}$. Also $(R\sin\phi)/(R\cos\phi) = 2/3$, and thus $\tan\phi = 2/3$, $\phi = \tan^{-1}(2/3) = \arctan(2/3) = 0.588$ radians $= 33.7°$.

Let's end with a physics example.

Example 2.7:

From astronomical data tables (e.g. `http://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html`) we know that we can we can observe an apparent diameter of the planet mars between 3.5 and 25.7 arcseconds. Given the radius of the planet (3390 km), evaluate the distance of closest approach as well as the largest distance to earth.

Solution:

This is a simple trig problem, and it helps (as always) to draw a picture, see Fig. 2.6. From that picture we see that with distance d, radius r, the angle under which we see mars satisfies $\tan(\phi/2) = d/R$. Actually, for the small angles under consideration $\tan(x) = x$, if we express x in radians. Thus

$$d = 2R/phi \quad .$$

Realizing there are 3600 arcseconds in a degree (60 second in a minute, 60 minutes in a degree), we find that we find that

$$d = 2R3600180/(\pi phi) \quad .$$

Substituting the values given we find a distance of closest approach of 6.38859×10^7 km and a largest distance of 4.69105×10^8 km

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Figure 2.6: The angle under which we see Mars.

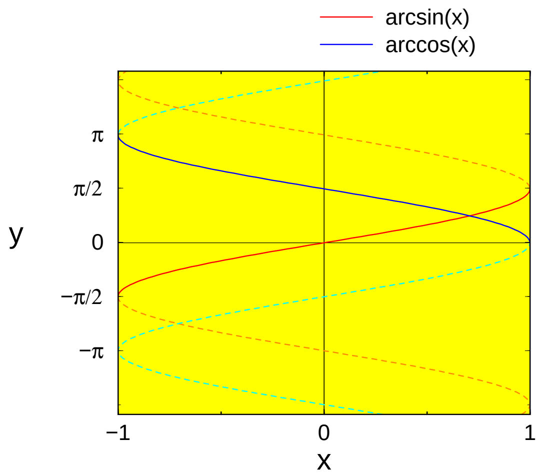


Figure 2.7: A plot of the inverse of the sine and cosine.

2.2.2 Inverse Trig Functions

L&T, 1.5.3.3

arcsin

The two alternative forms $y = \sin^{-1}(x)$ or $y = \arcsin(x)$ indicate that y is an angle whose sine is x.
Example 2.8:

Find $\sin^{-1}(1)$ and $\sin^{-1}(1/2)$.

Solution:

$y = \sin^{-1}(1)$ means $\sin(y) = 1$. Therefore $y = 90^\circ = \pi/2$ rads.
$y = \sin^{-1}(1/2)$ means $\sin(y)) = 1/2$, and thus $y = 30^\circ = \pi/6$ rads.

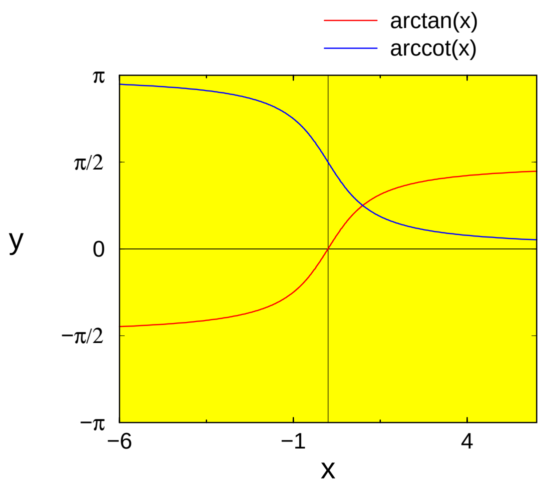


Figure 2.8: A plot of the inverse tangent and cotangent.

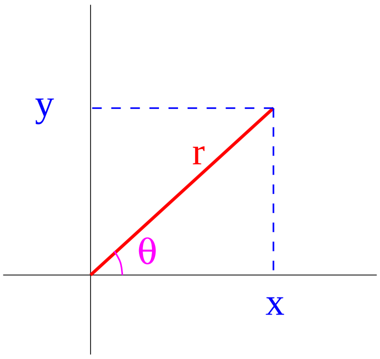


Figure 2.9: The meaning of polar coordinates.

Note: $\sin 30° = 1/2$, and $\sin 150° = 1/2$, and $\sin 390° = 1/2$, etc., so $\sin^{-1}(x)$ is a *multivalued* function. We need extra information, e.g., from the engineering situation or common sense to say which angle we are looking at.
The equation $y = \sin^{-1}(x)$ means the same as $x = \sin y$, (graph of $y = \sin x$ but with axis switched), note $-1 \leq x \leq 1$.
Note: $\sin^{-1}(x)$ is not the same as $\frac{1}{\sin x} = \sin(x)^{-1}$! The notation is very poor here but unfortunately very widely used. $\arcsin x$ would be better but not too common!

arccos

Similarly $y = \cos^{-1} x = \arccos x$ means $\cos y = x$. Once again, $-1 \leq x \leq 1$.

arctan

$y = \tan^{-1} x = \arctan x$ means $\tan y = x$.
Example 2.9:

Find x given $2\cos x = \sin x$.

Solution:

Divide by $\cos x$: $2 = \tan x$, or $x = \tan^{-1}(2)$.

2.3 Polar Coordinates

L&T, 1.9.3.3

The position of any point P in two-dimensional space can be specified by giving its (x, y) coordinates. However we could also say where P is by giving the distance from the origin 0, and the direction we need to go.

These two quantities are the polar coordinates (r, θ) of P. From a right angled triangle we see that $r\cos\theta = x$, and $r\sin\theta = y$, so $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$, and thus $r = \sqrt{x^2 + y^2}$. (N.B. We always take positive square root here!) Also $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$, Therefore $\theta = \tan^{-1}(y/x)$. In this case we must always draw a diagram. The reason is that two different angles can have the same tangent. The only relevant once for polar coordinates are that $\tan\theta_1 = \tan\theta_2$, when $\theta_2 = 180° + \theta_1 = \pi + \theta_1$. If P is in first or second quadrant we use θ_1, and if P is in third or fourth quadrant we use θ_2. So always draw a little sketch!
Example 2.10:

Find the polar coordinates corresponding to $x = -1, y = -1$.

Solution:

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$, and $\tan\theta = y/x = \frac{-1}{-1} = 1$. From the sketch we see that $\theta = 225° = \frac{5\pi}{4}$

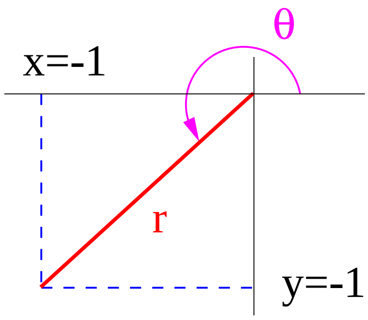


Figure 2.10: A sketch of (-1,-1) and polar coordinates.

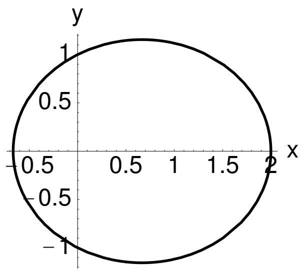


Figure 2.11: The Keplerian ellipse obtained for $\epsilon = -1/2$.

2.3.1 Polar curves

Often we wish to draw curves in polar coordinates; the most important example are the Kepler orbits, the ones resulting from a particle moving in the gravitational fiels of a single orbit, e.g., a single planet/comet orbiting the sun.

The Kepler orbits can be shown to take the form

$$r^{-1} = R_0^{-1}(1 + \epsilon \cos(\phi - \phi'))$$

Here R_0 is a quantity with unit length, determined from masses and gravitational parameters. We now use this relation (with $\phi' = 0$, for simplicity) to find the typical orbits for $\epsilon = 0$, $|\epsilon| < 1$ (we shall choose $-1/2$), $|\epsilon| = 1$, and $|\epsilon| > 1$ (we shall choose 2).

In order to plot these results we rewrite the relation as

$$r/R_0 = 1/(1 + \epsilon \cos\phi) \quad ,$$

and plot the value of r for each ϕ (or a suitably chosen selection).

$\epsilon = 0$

This is a circle.

$\epsilon = -1/2$

In this case it is not very hard to solve the problem: All values of ϕ give a positive r, and the easiest solution is just to plot a suitable large number of values. Obvious choices are $\phi = 0, \pi/6, \pi/4, \pi/3, \pi/2, \ldots$, and these immediately lead to the elliptical structure shown in Fig. 2.11. It can be shown that this is a real ellipse, with the origin (the sun around which the palnet revolves) as one of the focusses of the ellipse.

$\epsilon = 1$

In this case we cannot use $\phi = \pi$, and we thus conclude that the curve moves away to infinity. Once again we can draw a large number of points and we find a parabola, see Fig. 2.12.

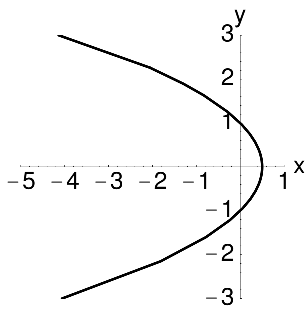


Figure 2.12: The Keplerian parabola obtained for $\epsilon = 1$.

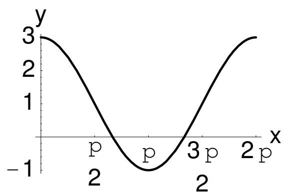


Figure 2.13: The range where r is positive for $\epsilon = 2$.

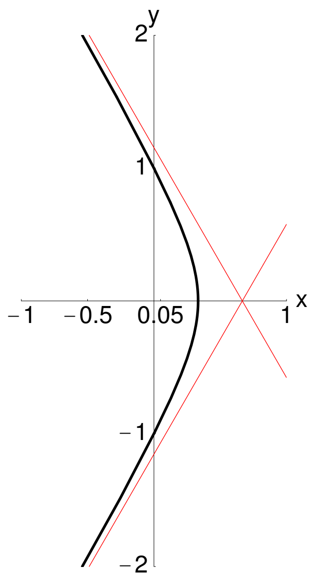


Figure 2.14: The Keplerian hyperbola obtained for $\epsilon = 2$.

$\epsilon = 2$

We need to carefully find the allowed range for ϕ, see Fig. 2.13, and we conclude that $-2\pi/2 < \phi < 2\pi/3$. Near the end points r diverges, and we can actually expand the value of r in the behaviour near these two points (Callenge question: how?) to find the two asymptotes $y = \pm\left(2/\sqrt{3} - \sqrt{3}x\right)$, which as we can see from Fig. **??** are indeed correct. The curve obtained is a hyperbola.

Chapter 3

Vectors in 2-space and 3-space

3.1 solid geometry

In a 3-dimensional world we have to consider 3-dimensional coordinate geometry rather than 2-dimensional.

First of all we set up a set of 3 mutually orthogonal coordinate axes, usually labeled x, y and z, see Fig. 3.1.

The z of axis is called right-handed, using the cork-screw rule: when rotating from x to y the z-axis is in the up direction. We can specify any point p by its coordinates (x, y, z). From 2D geometry we know that $OQ^2 = x^2 + y^2$. Thus

$$OP^2 = OQ^2 + z^2 = x^2 + y^2 + z^2.$$

If we call, as is conventional, $OP = r$, then

$$\boxed{r^2 = x^2 + y^2 + z^2.}$$

3.2 Vectors and vector arithmetic

3.2.1 What is a vector?

In order to understand what a vector is we must distinguish carefully between:

Scalars: These are specified by (i) their units, and (ii) the number of units. Together we refer to this as their magnitude. Examples are length, density, time, temperature, speed, etc.

Vectors: These are specified by (i) their units, (ii) the number of units and (iii) a direction. Examples are velocity, acceleration, force, heat flux, etc.

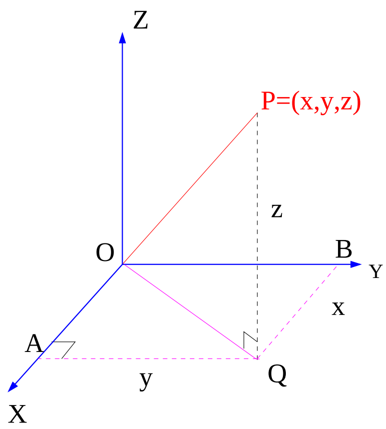


Figure 3.1: 3d geometry

A short word about notation: we shall use the notation $\overrightarrow{AB}$ for a vector pointing from A to B, and $\mathbf{a}$ for an abstract vector. *These notations do not agree with Stroud, but are standard practice!* For handwriting, where we cannot write a boldface letter, we shall use an underline ($\underline{a} = \mathbf{a}$) to denote the boldface.

3.2.2 Graphical representation

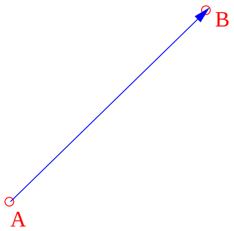


Figure 3.2: A vector represented by a directed line segment

We often represent a vector by a line-segment pointing from a point A to a point B, so that it has both direction an length, see Fig. 3.2. The length of the segment AB gives the magnitude and the arrow specifies the direction. The vector $\overrightarrow{AB}$ is often called a **displacement vector**, since, unlike an abstract vector, it has a begin- and end-point. We say that the displacement vector $\overrightarrow{AB}$ represents the abstract vector $\mathbf{a}$ if the direction and magnitude agree.

3.2.3 Equality and line of action

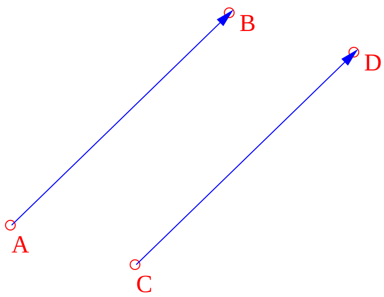


Figure 3.3: The representatives of two equal vectors.

Two vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal if they have the same magnitude (including units!) and direction, even if their representatives do not act along the same line, see Fig. 3.3: A vector can be moved parallel to itself without changing its value.

The line along which the vector points is called the *line of action*.

3.2.4 Negative of a vector

We shall often use the negative of a vector. The vector $-\mathbf{F}$ is defined as a vector of the same magnitude as $\mathbf{F}$, but pointing in the opposite direction, see Fig. 3.4. If $\mathbf{F}$ is represented by $\overrightarrow{AB}$, or loosely (i.e., equality denotes

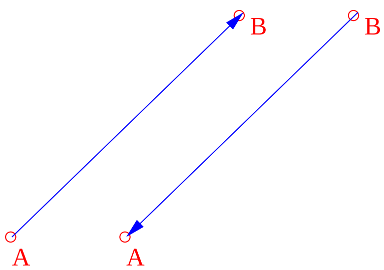


Figure 3.4: The negative of a vector.

"is represented by") $\mathbf{F} = \overrightarrow{AB}$, then $-\mathbf{F} = \overrightarrow{BA}$.

3.2.5 magnitude of a vector

We use as special notation for the magnitude: AB, $|\overrightarrow{AB}|$ or $|\mathbf{a}|$ or a. This is a scalar describing the length of the vector, and is therefore always positive. It does carry the same units, however.

3.2.6 Multiplication by a scalar

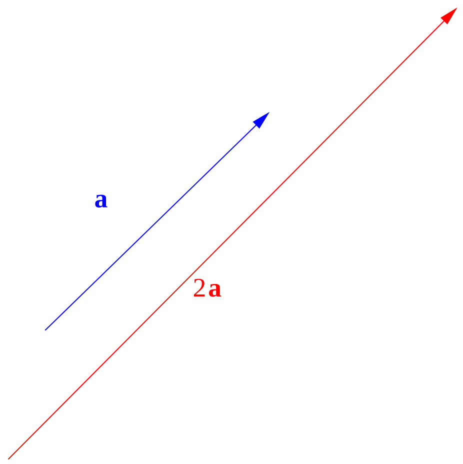


Figure 3.5: A vector, and twice the same vector.

Given a scalar a and a vector $\mathbf{F}$, then $a\mathbf{F}$ is a vector of the magnitude $|a|F$ and the same direction as $\mathbf{F}$ if a is positive, and oposite to $\mathbf{F}$ if a is negative, see Fig. 3.5. Thus $1 \cdot \mathbf{F} = \mathbf{F}$, $-1 \cdot \mathbf{F} = -\mathbf{F}$.

3.2.7 Unit vectors

Unit vectors have magnitude 1 (they are dimensionless, i.e., mathematical objects). We often define unit vectors associated with a physical vector. If $\mathbf{n}$ is a unit vector in the direction of a vector $\mathbf{F}$, then, using our laws of mutiplication, the vector $\mathbf{F} = F\mathbf{n}$, since the factors on both sides have the same direction and magnitude. From this we learn that

$$\boxed{\mathbf{n} = \frac{\mathbf{F}}{F}} ,$$

a relation used frequently.

3.3 Vector Addition

Addition of vectors is achieved by moving the starting point of the second vector to coincide with the endpoint of the first.

3.3.1 Triangle Law

Thus, as shown in Fig. 3.6 the displacement vectors are aligned, and we have $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. If the displacements represent $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, respectively we see that $\mathbf{a} + \mathbf{b} = \mathbf{c}$, or "changing sides" $\mathbf{c} = \mathbf{a} + \mathbf{b}$. This is called the triangle law of addition. It is used by always drawing displacement vectors that connect in the order of the addition. I.e., in the addition above the endpoint of the representative of $\mathbf{a}$ coinides with the start point of the vector $\mathbf{b}$. The sum vector is often called the *resultant.*

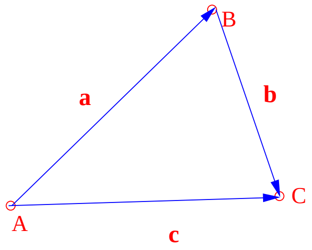


Figure 3.6: Addition of two vectors

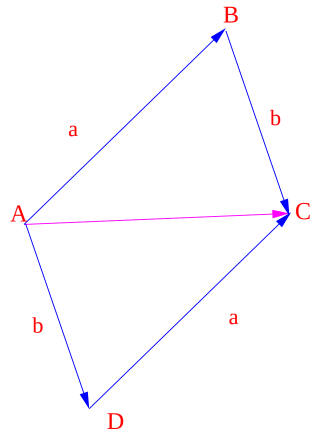


Figure 3.7: order of addition of two vectors

3.3.2 Parallelogram Law

If we investigate both $\mathbf{a}+\mathbf{b}$ and $\mathbf{b}+\mathbf{a}$, as shown in Fig. 3.7, we discover that the displacement vectors form the four sides of a parallelogram (parallelogram law), as well as the fact that the order of addition doesn't matter (commutativity):

$$\boxed{\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \quad .}$$

3.3.3 General Addition

If we wish to add several vectors, we repeat the procedure sketched for two vectors, putting all of them end to beginning, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{AF}$.

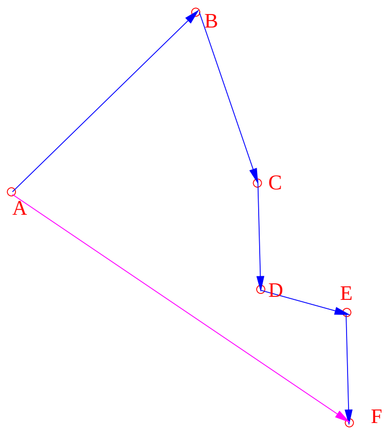


Figure 3.8: Addition of several vectors

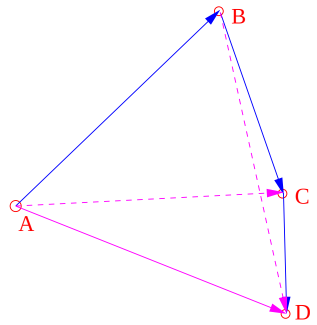


Figure 3.9: Associativity of addition

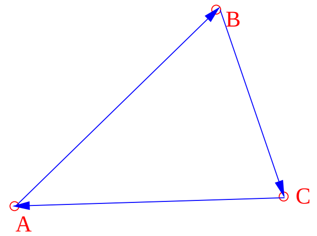


Figure 3.10: a closed set of vectors

3.3.4 Associativity

For number we know that they have the associative property, $(a + b) + c = a + (b + c)$. Let us investigate graphically whether such a relation holds for vectors. As we see from Fig. 3.9, this can be written in terms of displacement vectors as $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{CD}$, an obvious truth.

3.3.5 Closed sets of vectors: null vector

If we add together a set of vectors that returns to the starting point (a closed set of vectors), see Fig. 3.10, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \mathbf{0}$, we get a zero length vector (the null vector, see below).

3.3.6 Subtraction of vectors

If we subtract two vectors, we reverse the one with the minus sign (i.e., reverse the direction of the arrow on that vector) and use the rules for addition, $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

3.3.7 Zero or Null Vector

In subtraction if $\mathbf{b} = \mathbf{a}$ then $\mathbf{a} - \mathbf{a} = \mathbf{0}$ (zero or null vector). All null vectors are regarded as equal with zero magnitude but no natural direction. $\mathbf{0} + \mathbf{a} = \mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector $\mathbf{a}$.

3.4 Vectors: Component Form

3.4.1 Components in 2 dimensions

We look at a general vector $\mathbf{r} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC}$, see Fig. 3.11, which is decomposed into the sum of two vectors along the x and y axes. We define $\mathbf{i}$ as a unit vector in the x-direction, and $\mathbf{j}$ as a unit vector in y-direction. So $\overrightarrow{OA} = x\mathbf{i}$, $\overrightarrow{OB} = y\mathbf{j}$. Thus

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \qquad |\mathbf{r}| = \sqrt{(x^2 + y^2)},$$

where x and y are the components of $\mathbf{r}$ in the x and y directions. The vecor $\mathbf{r}$ as represented by the vector $\overrightarrow{OB}$ is called a *coordinate vector*.

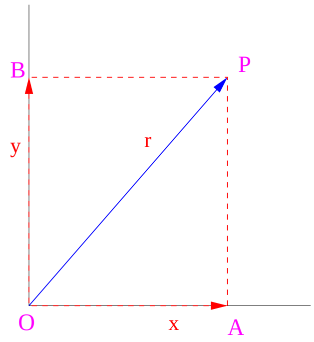


Figure 3.11: Components of a vector in two dimensions.

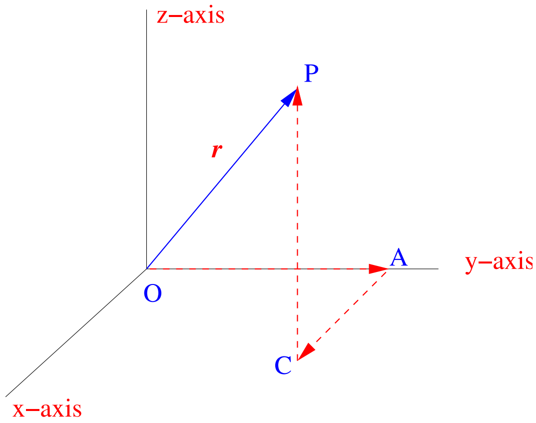


Figure 3.12: Components of a vector in three dimensions.

3.4.2 Vectors in 3 dimensions

As shown in Fig. 3.12, the result in three dimensions is quite similar. Let **i**, **j**, **k** be the right-handed set of unit vectors in the x, y, z direction, respectively.[1] Thus

$$\begin{aligned}\mathbf{r} &= \overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP} = \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CP} \\ &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k},\end{aligned}$$

where x, y and z are the components of **r**.

We shall often use the notation (p_1, p_2, p_3) for a vector $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} = p_3\mathbf{k}$. Once again the vectors **r** and **p** were given as *position vectors*, the displacement vector for the point P starting from the origin. Using pythagoras' theorem repeatedly we see that $r^2 = x^2 + y^2 + z^2$, and thus $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

3.4.3 Sum and Difference of vectors in Component Form

Let

$$\begin{aligned}\mathbf{r}_1 &= x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \quad , \\ \mathbf{r}_2 &= x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k} \quad ,\end{aligned}$$

then

$$\begin{aligned}\mathbf{r}_1 + \mathbf{r}_2 &= (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k} \quad , \\ \mathbf{r}_1 - \mathbf{r}_2 &= (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} + (z_1 - z_2)\mathbf{k} \quad .\end{aligned}$$

(please verify these geometrically for 2 dimensional space)

[1] A set of vectors is called right handed if, when turning a corkscrew from the first to the second vector, it moves in the direction of the third.

Example 3.1:

Given the points $A = (1, -4, 2)$ and $B = (2, 2, -3)$, find the component from for the vector $\overrightarrow{AB}$.

Solution:

We realise that $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$, or, $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$. We thus find that $\overrightarrow{AB} = (\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = (1-2)\mathbf{i} + (-4+2)\mathbf{j} + (2+1)\mathbf{k} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

3.4.4 Unit vectors

We study $\overrightarrow{OP} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $\mathbf{r} = (x, y, z)$, $|\mathbf{r}| = r$.Then the unit vector in the direction of $\mathbf{r}$ is

$$\hat{\mathbf{r}} = \mathbf{r}/r = x/r\mathbf{i} + y/r\mathbf{j} + z/r\mathbf{k},$$

Clearly $|\hat{\mathbf{r}}| = \frac{x^2+y^2+z^2}{r^2} = 1$.

Example 3.2:

If $\mathbf{r} = 8\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ find r, $\hat{\mathbf{r}}$ and the direction cosines (dc's) of $\mathbf{r}$.

Solution:

$$r = |\mathbf{r}| = \sqrt{8^2 + 4^2 + (-1)^2} = \sqrt{81} = 9 \quad ,$$
$$\hat{\mathbf{r}} = \mathbf{r}/r = 8/9\mathbf{i} + 4/9\mathbf{j} - 1/9\mathbf{k} \quad .$$

The d.c's are the components of $\hat{\mathbf{r}}$, i.e., $l = 8/9$, $m = 4/9$, $n = -1/9$.

3.4.5 Scaling of Vector

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and λ and μ are scalars, then

$$\lambda\mathbf{a} + \mu\mathbf{b} = (\lambda a_1 + \mu b_1)\mathbf{i} + (\lambda a_2 + \mu b_2)\mathbf{j} + (\lambda a_3 + \mu b_3)\mathbf{k} \quad .$$

Example 3.3:

If $\mathbf{a} = 2\mathbf{i} - 7\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, find

(i) $2\mathbf{a}$,

(ii) $-3\mathbf{b}$,

(iii) $3\mathbf{a} - \mathbf{b}$, and

(iv) the unit vector in the direction of$\mathbf{a}$.

Solution:

(i) $2\mathbf{a} = 4\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}$,

(ii) $-3\mathbf{b} = -9\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}$,

(iii) $3\mathbf{a} - \mathbf{b} = 6\mathbf{i} - 21\mathbf{j} + 3\mathbf{k} - (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 3\mathbf{i} - 23\mathbf{j} + 8\mathbf{k}$,

(iv) $\hat{\mathbf{a}} = \frac{\mathbf{a}}{a} = \frac{2\mathbf{i}-7\mathbf{j}+\mathbf{k}}{\sqrt{(4+49+1)}} = \frac{1}{\sqrt{54}}(2\mathbf{i} - 7\mathbf{j} + \mathbf{k})$

Example 3.4:

Given the points $A = (5, -2, 3)$ and $B = (2, 1, -2)$ find: (i) The position vectors of A and B relative to the origin
(ii) the vector $\overrightarrow{AB}$,
(iii) the position vector of the mid-point P of AB.

Solution:

(i) $\overrightarrow{OA} = \mathbf{a} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
(ii) $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a} = -3\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$
(iii) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{7}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$

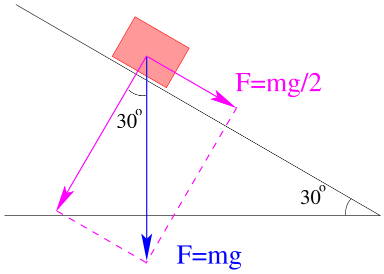


Figure 3.13: A truck on a slope

3.4.6 Physical example

Example 3.5:

A truck of mass 10000 kg stands on a slope that makes and angle of 30° with the horizontal.
1) Find the acceleration of the truck in terms of g.
2) An explosion exterts a force 10^5 N orthogonal to the surface. Find the reultant force (use

Solution:

1) Look at Fig. **??**. We see that the force parallel to the plave is $\frac{1}{2}mg$, orthogonal $\frac{1}{2}\sqrt{3}mg$. The acceleration is thus $\frac{1}{2}g$.
2) The new force, choosing the x axis parallel to the slope, and y orthogonal (upwards), is $(10^5 - \frac{1}{2}\sqrt{3}10^5)\mathbf{j} + 5 \times 10^4\mathbf{i} = 13397.5\mathbf{j} + 5 \times 10^4\mathbf{i}$. This has size 51763.8 N, and makes an angle of 15° with the slope, so 45° with the horizontal.

3.5 Vector products

We cannot easily generalise the product of two scalars to that of two vectors. We define new concepts of products as what has proven to be most useful in practice There are two types of product:

a) The scalar product, that takes two vectors and produces a scalar.

b) The vector product, that takes two vectors and produces a vector.

We shall take each of these in turn.

3.6 The scalar or dot product

The scalar product, also called dot product or inner product, of **a** and **b** is written as $\mathbf{a} \cdot \mathbf{b}$, and is defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta_{ab}, \tag{3.1}$$

This is clearly a number (scalar) and not a vector. The angle θ_{ab} is the angle between the first and second vector, and thus

$$\begin{aligned} \mathbf{b} \cdot \mathbf{a} &= |\mathbf{a}||\mathbf{b}| \cos\theta_{ba} \\ &= ab\cos(-\theta_{ab}) \\ &= ab\cos(\theta_{ab}) \\ &= \mathbf{a} \cdot \mathbf{b} \quad . \end{aligned}$$

(One usually suppresses the subscript ab on the angle θ.) We thus see that order does not matter, or more formally, that the dot product is commutative.

Let us look at some special cases

1. **a** is perpendicular to **b**. In that case $\theta = 90° = \pi/2$, and the cosine is zero: $\mathbf{a} \cdot \mathbf{b} = 0$.

2. $\mathbf{a}$ is parallel to $\mathbf{a}$, i.e., $\theta = 0$. $\mathbf{a}\cdot\mathbf{a} = a^2$. For that reason one sometimes writes $\mathbf{a}^2$ for a^2. Also

$$\boxed{a = \sqrt{\mathbf{a}\cdot\mathbf{a}} \quad .}$$

3. $$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \quad ,$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \quad .$$

   This is a straighforward application of the previous two properties! A set where each vector is orthogonal to all the others is called an *orthogonal set* of vectors; if the vectors also have unit length, one speaks of an *orthonormal set*.

It is generally useful to list a few more properties:

1. $(m\mathbf{a})\cdot\mathbf{b} = (ma)b\cos\theta = m(ab\cos\theta) = m\mathbf{a}\cdot\mathbf{b}$. (What is $(2\mathbf{a})\cdot(2\mathbf{b})$?)
2. $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ is the product of the *scalar* $\mathbf{a}\cdot\mathbf{b}$ with the vector $\mathbf{c}$. Thus the result has the same direction as $\mathbf{c}$, with magnitude $(\mathbf{a}\cdot\mathbf{b})c$.
3. We can divide by $\mathbf{a}\cdot\mathbf{b}$ since it is a scalar! (Conversely, we can *not* divide by a vector!)
4. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$. (Distributive law). This will not be proven here, but can easily be done using component form.

Example 3.6:

Simplify $(\mathbf{a}+\mathbf{b})^2$

Solution:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\ &= (\mathbf{a}+\mathbf{b})\cdot\mathbf{a} + (\mathbf{a}+\mathbf{b})\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}\\ &= a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

3.6.1 Component form of dot product

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\cdot(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\ \mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3 \quad .\end{aligned}$$

Example 3.7:

Find a unit vector which is perpendicular to $(1, 2, -1)$ and has y-component zero.

Solution:

This vector has the form $\mathbf{a} = (a_x, 0, a_z)$. Must be orthogonal to $(1, 2, -1)$, so

$$(a_x, 0, a_z)\cdot(1, 2, -1) = 0,$$

which leads to

$$a_x - a_z = 0, \qquad a_x = a_z = \alpha.$$

For this to be a unit vector $a^2 = 2\alpha^2 = 1$, or $\alpha = \pm 1/\sqrt{2}$ (we can choose either sign. Explain!). Thus

$$\boxed{\mathbf{a} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}) \quad .}$$

3.7 Angle between two vectors

Let $\mathbf{a}$ and $\mathbf{b}$ include the angle θ. By definition $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. Thus $\cos\theta = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}||\mathbf{b}|$, or $\cos\theta = \mathbf{a}/|\mathbf{a}|\cdot\mathbf{b}/|\mathbf{b}|$, or $\cos\theta = \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$. Thus $\cos\theta$ is the dot product of the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

Example 3.8:

Consider the vectors $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (1, 1, 2)$. Find $\mathbf{u}\cdot\mathbf{v}$ and determine the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution:

First Calculate

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = (2)(1) + (-1)(1) + (1)(2) = 3.$$

Also $|\mathbf{u}| = \sqrt{6}$ and $|\mathbf{v}| = \sqrt{6}$, so

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}.$$

Hence $\theta = \frac{\pi}{3}$ (or $60°$).

3.8 Work

In mechanics the work performed by a force is defined as the product of the magnitude of the force times the distance moved in the direction of the force.

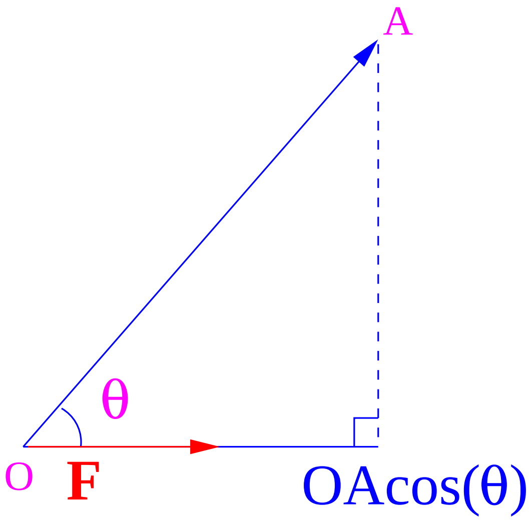


Figure 3.14: The work done by the force $\mathbf{F}$ if a mass moves from O to A equals $OA\cdot F\cdot\cos\theta$.

From Fig. 3.14 we see that, since the component of OA along the line of force is $OA\cos\theta$, where OA is the distance d travelled, the work is $W = d\,\cos\theta\,F = \mathbf{d}\cdot\mathbf{F}$, and thus work can be evaluated as an innerproduct.

Example 3.9:

A force $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ N is applied to a particle which is moving along a wire OAB where OA and AB are straight, and the points A and B are $A = (1, 0, 0)$ m and $B = (2, 2, -2)$ m. Find the work done.

Solution:

Along the line OA the work done is $\mathbf{F}\cdot\overrightarrow{OA}$,

$$W_1 = (2, 3, -1)\cdot(1, 0, 0) = 2 \text{ J}.$$

Along the line AB, $\overrightarrow{AB} = (1, 2, -1)$, and the work done is

$$W_2 = (2, 3, -1)\cdot(1, 2, -1) = 2 + 6 + 1 = 9 \text{ J}.$$

The total work is thus

$$\boxed{W = 2 + 9 = 11 \text{ J}.}$$

3.9 The vector product

We have now looked extensively at the scalar product, and now look at the vector product, that returns a vector. Two standard notations are used

$$\mathbf{a} \times \mathbf{b}, \text{and } \mathbf{a}\hat{}\mathbf{b}. \tag{3.2}$$

We shall use the first notation. Other terms used are "cross product" or "outer product".

The vector product of two vectors **a** and **b** is defined as a vector, see Fig. 3.15,

- of magnitude $ab|\sin\theta|$
- of a direction orthogonal to both **a** and **b**, so that **a**, **b** and $\mathbf{a} \times \mathbf{b}$ form a right-handed set

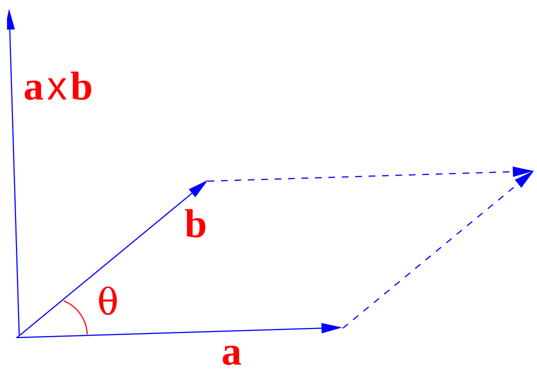


Figure 3.15: The definition of the outer product.

The magnitude of the outer product is exactly equal to the area of the parallelogram with sides **a** and **b**, $A = ab\sin\theta$. calculation of the outer product in component form (to be discussed below) is thus an easy way to obtain this area.

Let **n** be a unit vector in the direction of $\mathbf{a} \times \mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = ab\sin\theta\mathbf{n}$. From the right handed rule we see that $\mathbf{b} \times \mathbf{a} = ab\sin\theta(-\mathbf{n}) = -\mathbf{a} \times \mathbf{b}$, i.e., the vector product is *not commutative*. Properties of the outer product:

1. For parallel vectors $\theta = 0$ and so $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, in particular $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
2. For orthogonal vectord, i.e., the angle θ between **a** and **b** is $\pi/2$, any two of the vectors **a**, **b** and $\mathbf{a} \times \mathbf{b}$ are orthogonal.
3. The coordinate vectors **i**, **j**, **k**:
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.$$
$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}. \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}. \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$
4. From $\mathbf{a} \times \mathbf{b} = ab\sin\theta\mathbf{n}$ we see that $(n\mathbf{a}) \times \mathbf{b} = (ma)b\sin\theta\mathbf{n} = m(\mathbf{a} \times \mathbf{b})$.
5. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$. Follows most easily from component form (see below).
6. Component form:
Using $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and similar for $vecb$, we find
$$\begin{aligned} \mathbf{c} =& \mathbf{a} \times \mathbf{b} \\ =& (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ =& a_xb_x\mathbf{i} \times \mathbf{i} + a_xb_y\mathbf{i} \times \mathbf{j} + a_xb_z\mathbf{i} \times \mathbf{k} + \\ & a_yb_x\mathbf{j} \times \mathbf{i} + a_yb_y\mathbf{j} \times \mathbf{j} + a_yb_z\mathbf{j} \times \mathbf{k} + \\ & a_zb_x\mathbf{k} \times \mathbf{i} + a_zb_y\mathbf{k} \times \mathbf{j} + a_zb_z\mathbf{k} \times \mathbf{k} \\ =& \mathbf{i}(a_yb_z - a_zb_y) + \mathbf{j}(a_zb_x - a_xb_z) + \mathbf{k}(a_xb_y - a_yb_x) \end{aligned}$$

This last line is often summarized in the form of a determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}.$$

Example 3.10:

Give $\mathbf{a} = (6, 1, 3)$ and $\mathbf{b} = (-2, 0, 4)$, find $\mathbf{a} \times \mathbf{b}$.

Solution:

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(1 \cdot 4 - 3 \cdot 0) + \mathbf{j}(3 \cdot (-2) - 6 \cdot 4) + \mathbf{k}(6 \cdot 0 - (-1) \cdot (-2)) = (4, -30, 2).$$

Example 3.11:

Find $\mathbf{a}\times\mathbf{b}$ given $\mathbf{a} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, and $\mathbf{b} = 2\mathbf{i}-\mathbf{j}+\mathbf{k}$,and find $\hat{\mathbf{n}}$ the unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 2 & -1 \\ \mathbf{k} & -1 & 1 \end{pmatrix}.$$

Expand by Row 1: and we get $\mathbf{i}(2 - 1) - \mathbf{j}(1 + 2) + \mathbf{k}(-1 - 4) = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$.

$$\begin{aligned} \hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} &= \frac{\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}}{\sqrt{1 + 9 + 25}} \\ &= \frac{1}{\sqrt{35}}(\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}) \end{aligned}$$

Other examples:

3.10 *triple products*

The inneer product $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and we can't use the result in further vector or dot products. The outer product $\mathbf{a}\times\mathbf{b}$ is a vector so it may be combined with a third vector $\mathbf{c}$ to form either a scalar product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, or a vector product: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

We shall look at the scalar triple product,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (|\mathbf{a}||\mathbf{b}|) \sin\theta \hat{\mathbf{n}} \cdot \mathbf{c}.$$

It is clearly a scalar quantity since $\hat{\mathbf{n}} \cdot \mathbf{c}$ is a number. It is particularly relevant to study the geometric interpretation, as in Fig. 3.16.

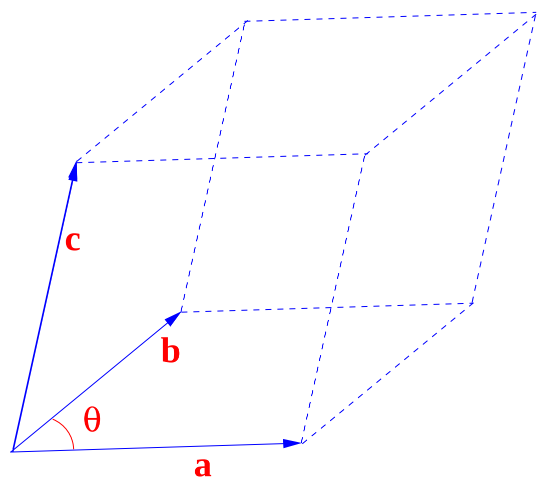


Figure 3.16: The parallelopipid related to the scalar triple product.

The quantity $\mathbf{n} \cdot \mathbf{c}$ is the height of the parallelopiped in that figure, adn we find that

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |(|\mathbf{a}||\mathbf{b}| \sin\theta)||\hat{\mathbf{n}} \cdot \mathbf{c}| = \text{Area of base } \times \text{Height} = V$$

where V is the volume of the parallelopiped. V is independent of the way it is calculated, i.e., any face may be used as base. Hence

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\ &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \end{aligned}$$

Since scalar product is commutative

$$\begin{aligned} (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} &= (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \end{aligned}$$

All the six expressions are equal! The $\cdot$ and the $\times$ may be interchanged as long as product is defined.

3.10.1 Component Form

We know that

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k},$$

then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ with $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$,

$$(\mathbf{a} \times \mathbf{b})\mathbf{c} = (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$$

This can be put in determinant form,

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Note that the order of the columns rows is the same as the order of the vectors. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the STP.
Example 3.12:

Find $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ given $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 3\mathbf{j} + \mathbf{k}$, $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

$$\det \begin{pmatrix} 1 & 0 & 1 \\ -2 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} + \det \begin{pmatrix} -2 & 3 \\ 0 & 1 \end{pmatrix} = -4 - 2 = -6$$

3.10.2 Some physical examples

Important physical quantities represented by a vector product are

- Angular momentum: This is defined as the product of the distance from a centre with the momentum perpendicular to this distance;

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}.}$$

- Magnetic force. The force on a charged particle (charge q) moving with velocity $\mathbf{v}$ in a constant magnetic field $\mathbf{B}$ is perpindicular to both $\mathbf{v}$ and $\mathbf{B}$, with size commensurate with the outer product

$$\boxed{\mathbf{F} = q\mathbf{v} \times \mathbf{B}.}$$

- Torque: The torque of a force describes the rotational effect of such a force (think about moving a crank). Clearly only the force perpendicular to the crank makes it rotate, hence the definition

$$\boxed{\mathbf{T} = \mathbf{r} \times \mathbf{F}.}$$

3.11 *Vector Triple Product*

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is perpendicular to both $\mathbf{c}$ and $\mathbf{a}\times\mathbf{b}$, so lies in the plane of $\mathbf{a}$ and $\mathbf{b}$. Basic result obtained easily, is,

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}.$$

NB. The order and the brackets must not be changed , if we do this will alter the result. If $\mathbf{c}$ is normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ then $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{0}$ (Why?)

Example 3.13:

Find $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ given $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}\\ &= 5\mathbf{b} + 3\mathbf{a}\\ &= 13\mathbf{i} + \mathbf{j} - 8\mathbf{k},\end{aligned}$$

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\\ &= -[(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}]\\ &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\\ &= 5\mathbf{b} - \mathbf{c}\\ &= 9\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}\end{aligned}$$

3.12 *The straight line*

Straight line through A (with position vector $\mathbf{a}$) and parallel to a vector $\mathbf{b}$. Let P be a general point on L, then $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{r} = \mathbf{a} + \overrightarrow{AP}$. Since $\overrightarrow{AP}$ is parallel to $\mathbf{b}$, hence $\overrightarrow{AP} = \lambda\mathbf{b}$ (for some scalar λ), λ may be positive or negative. Thus $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. This is the vector equation of a straight line.

3.12.1 Standard form of L

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ the equation

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},$$

gives $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (a_1 + \lambda b_1)\mathbf{i} + (a_2 + \lambda b_2)\mathbf{j} + (a_3 + \lambda b_3)\mathbf{k}$. Equality of the vectors gives 3 scalar equations, $x = a_1 + \lambda b_1$ or $\frac{(x-a_1)}{b_1} = \lambda$, $y = a_2 + \lambda b_2$ or $(y - a_2)/b_2 = \lambda$ and $z = a_3 + \lambda b_3$ or $(z - a_3)/b_3 = \lambda$. Since $-\infty < \lambda < \infty$, (for different points on L), we find that these three scalar equations give the Cartesian equations of L as

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda$$

This is called the standard or canonical form.

In standard form:

(i) Equating numerators to zero determines a point on L (i.e., A).

(ii) Denominators give the direction ratios of L (i.e., the direction of the vector $\mathbf{b}$)

Example 3.14:

Find the position vector of a point on a straight line L and a vector along L whose Cartesian equations are $\frac{3x+1}{2} = \frac{y-7}{3} = \frac{-2z+1}{4}$.
the standard form of L is

$$\frac{x + \frac{1}{3}}{\frac{2}{3}} = \frac{y - 7}{3} = \frac{z - \frac{1}{2}}{-2}$$

Point A: $\left(-\frac{1}{3}, 7, \frac{1}{2}\right)$, position vector of A: $-\frac{1}{3}\mathbf{i} + 7\mathbf{j} + \frac{1}{2}\mathbf{k}$. $\mathbf{b} = \frac{2}{3}\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ (parallel to L)

Example 3.15:

Example: Find the Cartesian equations of a straight line L through the point $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in the direction of the vector $\mathbf{b} = -2\mathbf{j} + 3\mathbf{k}$.
L: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ gives $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(0.\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$. This gives the following Cartesian equations of L:

$$\frac{x-1}{0} = \frac{y+2}{-2} = \frac{z-1}{3} (= \lambda)$$

Chapter 4

Differentiation

L&T, F.9

4.1 Assumed knowledge

4.1.1 First principles definition

If $y = f(x)$ and x increases from x to $x + \delta x$ then the change in y is give by $\delta y = f(x + \delta) - f(x)$, see Fig. 4.1. The differential is defined as

$$\boxed{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.}$$

4.1.2 Meaning as slope of a curve

The derivative can also be interpreted as the slope of a curve, see Fig. 4.2. If the slope at a given point has an angle θ, we find that $\tan\theta$ is $\frac{dy}{dx}$. In other words, the line $y - y_0 = \tan\theta\,(x - x_0)$ is tangent to the curve at (x_0, y_0).

4.1.3 Differential of a sum

The differential of a sum is the sum of differentials,

$$\boxed{\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.}$$

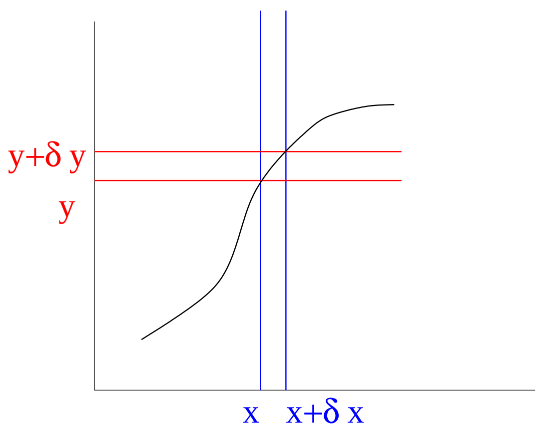


Figure 4.1: The definition of the differential.

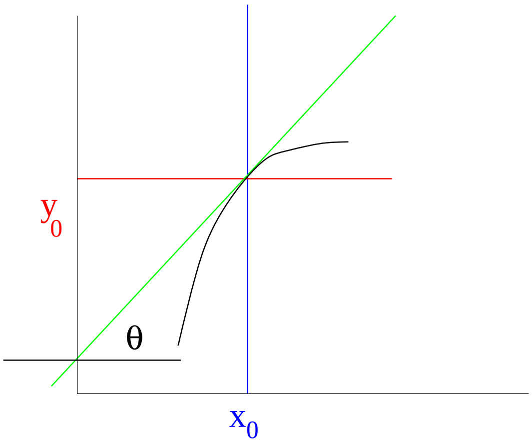


Figure 4.2: The definition of the differential.

4.1.4 Differential of product

L&T, F.9.26-27

There exists a simple rule to calculate the differential of a product,

$$\boxed{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.}$$

E.g., if $y = x^2 \sin x$,

$$\frac{dy}{dx} = x^2 \cos(x) + 2x \sin(x) \quad .$$

4.1.5 Differential of quotient

L&T, F.9.28-30

In the same way we can find a relation for the differential of a quotient,

$$\boxed{\frac{d(\frac{u}{v})}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.}$$

E.g., if $y = \frac{\sin x}{x}$,

$$\frac{dy}{dx} = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \quad .$$

4.1.6 Function of a function (chain rule)

L&T, F.9.33-36,7.5-18

Often we take a function of a function. In such a case, where $y = f(g(x))$ we put $z = g(x)$, and find

$$\boxed{\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}.}$$

This rule is sometimes expressed in words as “the differative of the function, times the derivative of its argument”, and you may know it as

$$\boxed{\frac{dg(f(x))}{dx} = f'(g(x))g'(x).}$$

Example 4.1:

Find $\frac{dy}{dx}$ for $y = \cos(\ln x)$.

Solution:

Put $z = \ln x$ so $y = \cos z$,

$$\frac{dy}{dx} = \frac{dy}{dz}, \quad \frac{dz}{dx} = -\sin z \frac{1}{x} = -\frac{\sin(\ln x)}{x} \quad .$$

Example 4.2:

Find $\frac{dy}{dx}$ for $y = \sin^3(2x-1)$.

Solution:

Put $z = \sin(2x-1)$ so $y = z^3$,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 3z^2 2\cos(2x-1) = 6\sin^2(2x-1)\cos(2x-1) \quad .$$

4.1.7 some simple physical examples

Example 4.3:

Given that $x(t) = 5t^2$ m, find the velocity $v(t)$ and the acceleration $a(t)$.

Solution:

Using the definitions of velocity as rate of change of position, we find that $v = \dot{x} = \frac{dx}{dt} = 10t$ m/s, and with accelartion as rate of change of velocity, we have $a = \dot{v} = \ddot{x} = \frac{dv}{dt} = 10$ m/s^2.

Example 4.4:

For simple harmonic motion (SHM) $x = \cos(\omega t)$. Find the velocity and acceleration.

Solution:

Use the change rule for differentiation, $v = \dot{x} = -\omega\sin(\omega t)$, $a = \dot{v} = -\omega^2\cos(\omega t)$

4.1.8 Differential of inverse function

L&T, 9.8-13

When we wish to calculate the differential of an inverse function, i.e, a function g such that $g(f(x)) = x$, we can use our knowledge of the derivative of f to find that of g.

Example 4.5:

Find the derivative of $y = \sin^{-1} x$.

Solution:

We use $y = \sin(x)$ and calculate $\frac{dx}{dy}$ first,

$$\frac{dx}{dy} = \frac{\sin y}{dy} = \cos y.$$

Now $\cos y = \pm\sqrt{1-\sin^2 y}$, but the slope of the inverse sine is always positive. Thus

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sqrt{1-x^2}}.$$

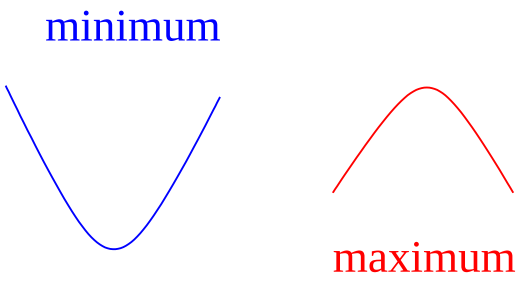


Figure 4.3: The meaning of a minimum and maximum.

4.1.9 Maxima and minima

L&T, 9.24-31

At a maximum or minimum the slope is 0 so that $\frac{dy}{dx} = 0$. To find which case it is, we look at $\frac{d^2y}{dx^2}$, which can easily be done from a plot of the slope.

Example 4.6:

Find all maxima and minima of $y = x(3 - x)$ and determine their character.

Solution:

We find that $\frac{dy}{dx} = x(-1) + (3 - x)1 = 3 - 2x$. For a maximum or minimum the slope must be 0. This happens for $3 - 2x = 0$, i.e., $x = \frac{3}{2}$. For that value of x, $\frac{d^2y}{dx^2} = -2$. So the point $x = 3/2$, $y = 9/4$ is a (and the only) maximum.

4.1.10 Higher Derivatives

L&T, F.9.21-22

Higher derivatives are obtained by differentiation 2 or more times, $\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx}$, $\frac{d^3y}{dx^3} = \frac{d(d^2y/dx^2)}{dx}$.

Example 4.7:

$y = \ln x$, $\frac{dy}{dx} = 1/x$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$, $\frac{d^3y}{dx^3} = \frac{2}{x^3}$, etc.

Example 4.8:

The equation for simple harmonic motion (SHM) is $\frac{d^2x}{dt^2} = -\omega^2 x$. Prove that $x = (A\cos\omega t) + B\sin\omega t$ satisfies this equation.

Solution:

We must differentiate twice, start with first derivative, $\frac{dx}{dt} = (-\omega)A\sin\omega t + \omega B\cos\omega t$, and find that

$$\begin{aligned} \frac{d^2x}{/}dt^2 &= -\omega^2 A\cos\omega - \omega^2 B\sin\omega t \\ &= -\omega^2(A\cos\omega t + B\sin\omega t) \\ &= -\omega^2 x. \end{aligned}$$

QED.

N.B.: SHM not studied here, but in next semester. The constants A, B can only be obtained with extra input.

4.2 Other techniques

4.2.1 Implicit Differentiation

L&T, 7.26-30

The equation of a circle $x^2 + y^2 = a^2$ is not in the form $y = f(x)$, (although it can be rearranged to $y = \pm\sqrt{a^2 - x^2}$). In this case it is easier to find $\frac{dy}{dx}$ directly without rearranging. Differentiate both sides of the equation $x^2 + y^2 = a^2$ with respect to x, assuming y to be a function of x. We find $2x + \frac{dy^2}{dx} = 0$. Now use $\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}$. (Proof: Put $z = y^2$ - need $\frac{dz}{dx}$, $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 2y\frac{dy}{dx}$.) So $2x + 2y\frac{dy}{dx} = 0$, or

$$\boxed{\frac{dy}{dx} = -\frac{x}{y} \quad .}$$

N.B.: This method usually gives $\frac{dy}{dx}$ in terms of both x and y.
Example 4.9:

Find $\frac{dy}{dx}$ for $x^2 + 4x + 3xy + y^3 = 6$.

Solution:

Differentiating both sides with respect to x we find

$$2x + 4 + 3y + 3x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \quad ,$$

we thus conclude that

$$\boxed{\frac{dy}{dx} = -\frac{(2x + 4 + 3y)}{(3x + 3y^2)} \quad .}$$

4.2.2 Logarithmic differentiation

L&T, 7.19-25

If a function has a large number of factors it may be easier to take the logarithm before differentiating, using the fat that the logarithm of a product is the sum of logarithms.
Example 4.10:

Find $\frac{dy}{dx}$ for $y = \frac{\sqrt{a+x}\sqrt{b-x}}{x-c}$.

Solution:

$$\ln y = \ln(\sqrt{a + x}) + \ln(\sqrt{b - x}) - \ln(x - c) = \frac{1}{2}\ln(a + x) + \frac{1}{2}\ln(b - x) - \ln(x - c) \quad .$$

Differentiate both sides with respect to x:

$$\frac{d \ln y}{dx} = \frac{1}{y}\frac{dy}{dx} \quad .$$

So

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{(a + x)} + \frac{1}{2}\frac{(-1)}{(b - x)} - \frac{1}{(x - c)} \quad .$$

and thus

$$\boxed{\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{(a + x)} - \frac{1}{(b - x)} - \frac{2}{(x - c)}\right)\frac{\sqrt{a + x}\sqrt{b - x}}{x - c} \quad .}$$

4.2.3 Differentiation of parametric equations

L&T, 7.31-36

Some equations can be written in parametric form, i.e., $x = x(t)$, $y = y(t)$ with t a parameter. We can then find its differential in terms of the parameter. We shall study this by means of an example only.

Example 4.11:

Given circle of radius 4,

$$x^2 + y^2 = 16 \tag{4.1}$$

use the parametric form to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(2\sqrt{3}, 2)$.

Solution:

The parametric form is

$$x = 4\cos\theta, \quad y = 4\sin\theta \quad ,$$

which clearly satisfies (4.1). Now

$$\boxed{\frac{dy}{dx} = \frac{dy}{df}\frac{df}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\cos\theta}{-4\sin\theta} = -\cot\theta.}$$

Note: result is in terms of θ. Then $y/4 = \sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$ (must be in first quadrant), and $\cos\theta = \frac{\sqrt{3}}{2}$ therefore $\frac{dy}{dx} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$. Now do $\frac{d^2y}{dx^2}$.

Note: $\frac{d^2y}{dx^2} \neq \frac{d^2y}{d\theta^2} / \frac{d^2x}{d\theta^2}$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} \\
&= \frac{d}{dx}(-\cot\theta) = \frac{d}{d\theta}(-\cot\theta)\frac{d\theta}{dx} \\
&= \frac{\operatorname{cosec}^2\theta}{dx/d\theta} = (\operatorname{cosec}^2\theta)/(-4\sin\theta) \\
&= -(1/4)\operatorname{cosec}^3\theta.
\end{aligned}$$

Other examples of parametric curves are

1. Ellipses $x^2/a^2 + y^2/b^2 = 1$: put $x = a\cos\theta$ and $y = b\sin\theta$,
2. Parabola $x^2/a^2 - y^2/b^2 = 1$: put $x = a\cosh\theta$ and $y = b\sinh\theta$.
3. Use of time t, e.g., for $x = 2t + 1$, $y = -gt^2/2 + 3t$.

4.3 Vector functions

In physical (epsically mechanics) problems we often have solutions in a form $\mathbf{r} = \mathbf{r}(t)$, a "vector function".
Example 4.12:

A particle moves along a circle with uniform angular frequency, $\mathbf{r} = \mathbf{i}\cos(\omega t) + \mathbf{j}\sin(\omega t)$. Find the velocity.

Solution:

If we are perfectly naive, we write $\mathbf{v} = \dot{\mathbf{r}} = -\mathbf{i}\omega\sin(\omega t) + \mathbf{j}\omega\cos(\omega t)$. This is actually correct!

The velocity is defined as the vector with as components the time-derivative of the components of the position vector,

$$\boxed{\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}.}$$

It is actually quite illustrative to look at a graphical representation of the procedure, see Fig. **??**. We notice there that the (vector) derivative of a vector function points is a vector that is tangent to (decsribes the local direction of) the curve: not a surprise since that is what velocity is!

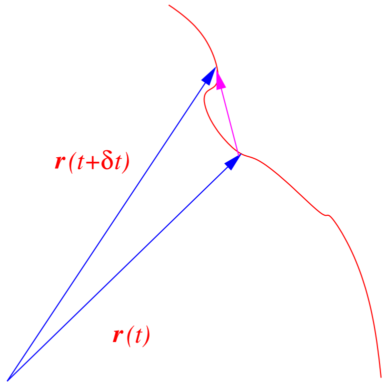


Figure 4.4: A schematic representation of the derivative of a vector function.

Example 4.13:

When a particle moves in a circle, find two independent way to show that $\mathbf{r}\cdot\dot{\mathbf{r}} = 0$.

Solution:

1) Use the uniform motion example from above, and we find $\mathbf{r}\cdot\mathbf{v} = -\omega\cos(\omega t)\sin(\omega t)+\omega\sin(\omega t)\cos(\omega t) = 0$. This is not a genral answer though!
2) Write $\mathbf{r}\cdot\mathbf{r} =$ constant. (Definition of circle!) Then, by differentiating both sides of the relation (in the "other" order), we find

$$\begin{aligned} 0 &= \frac{d\mathbf{r}\cdot\mathbf{r}}{dt} \\ &= \frac{dx^2+y^2+z^2}{dt} \\ &= 2x\frac{dx}{dt}+2y\frac{dy}{dt}+2z\frac{dz}{dt} \\ &= 2\mathbf{r}\cdot\dot{\mathbf{r}}. \end{aligned}$$

and we have the desired results.

Example 4.14:

Find the velocity of a partocle that moves from $\mathbf{r}_1 = (1, 2, 3)$ to $\mathbf{r}_2 = (3, 6, 7)$ in 2 s along a straight line with constant velocity. Also find the position 5 s after passing $\mathbf{r}_1$,

Solution:

Clearly $\mathbf{r} = \mathbf{r}_1 + \mathbf{v}t$ if the particle is at point 1 at $t = 0$, We get, substituting $t = 2$;

$$(3, 6, 7) = (1, 2, 3) + \mathbf{v}2,$$

from which we conclude (solving for each component separately) that $\mathbf{v} = (1, 2, 2)$. At time $t = 5$ we have

$$\mathbf{r} = (1, 2, 3) + (1, 2, 2)5 = (6, 12, 13).$$

4.3.1 Polar curves

Things get slightly more involved (but quite relevant!) when we look at curves in polar coordinates, i.e., specified by $r(t)$ and $\theta(t)$. From $\mathbf{r} = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j}$ we find that

$$\begin{aligned} \dot{\mathbf{r}} =&(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\mathbf{i} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\mathbf{j} \\ &= \dot{r}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + r\dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \qquad = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}. \end{aligned}$$

The first unitvector is indeed the one parallel to $\mathbf{r}$; the second one is defined from its expression. There is some interesting mathematics going on over here,

$$\hat{\mathbf{r}}\cdot\hat{\theta} = (\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\cdot(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = 0!$$

This is often used to say that r and θ are orthogonal coordinates, at each point they are associated with different, but always orthogonal directions!

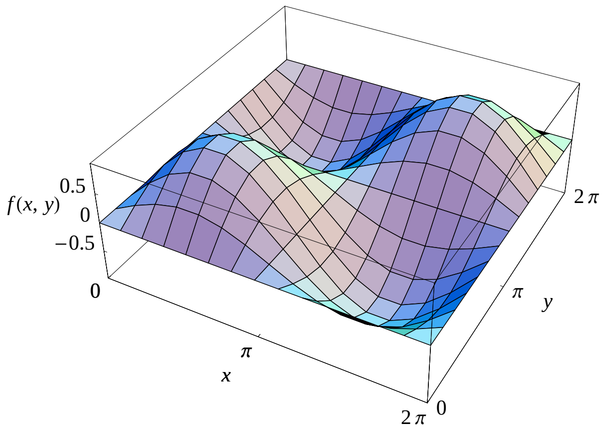


Figure 4.5: The surface $z = f(x, y) = \sin(x)\sin(y)$.

Example 4.15:

Express the velocity of a particle moving in an elliptic (Keppler) orbit,

$$r = \frac{1}{1 - \frac{1}{2}\cos(\theta)},$$

in turn of $\dot{\theta}$. Now calculate the kinetic energy of the particle.

Solution:

Obviously $\mathbf{r} = \frac{\cos\theta}{1-\frac{1}{2}\cos(\theta)}\mathbf{i} + \frac{\sin\theta}{1-\frac{1}{2}\cos(\theta)}\mathbf{j}$. Now differentiate w.r.t. t using the chain and quotient rules:

$$\begin{aligned}\mathbf{r} &= \dot{\theta}\left(\frac{-\sin\theta(1-\frac{1}{2}\cos(\theta)) - \cos\theta\frac{1}{2}\sin\theta}{(1-\frac{1}{2}\cos(\theta))^2}\mathbf{i} + \frac{\cos\theta(1-\frac{1}{2}\cos(\theta)) - \sin\theta\frac{1}{2}\sin\theta}{(1-\frac{1}{2}\cos(\theta))^2}\mathbf{j}\right)\\ &= \frac{\dot{\theta}}{(1-\frac{1}{2}\cos(\theta))^2}\left((-\sin\theta)\mathbf{i} + (\cos(\theta) - \tfrac{1}{2})\mathbf{j}\right).\end{aligned}$$

The kinetic energy is thus found to be

$$\begin{aligned}K = \tfrac{1}{2}mv^2 &= \tfrac{1}{2}m\frac{(\dot{\theta})^2}{(1-\frac{1}{2}\cos(\theta))^4}(\sin^2\theta + \cos^2\theta - \cos\theta + \frac{1}{4})\\ &= \tfrac{1}{2}m(\dot{\theta})^2\frac{\frac{5}{4} - \cos\theta}{(1-\frac{1}{2}\cos(\theta))^4}.\end{aligned}$$

4.4 Partial derivatives

In Figs. 4.5 and 4.6 we show an example of functions of more than one variable. Clearly it is very easy to pick out the minima and maxima, since we can make a very visual representation of such a function as a surface by the identification of the "height" z with the output of the function. In more than two dimensions, i.e., when we have a function that takes three or more arguments, and returns one value, we can't use the visual analogy. So how do we deal with that? We need to generalise derivatives to more than one dimension.

Let us study the situation in two dimensions, and generalise to three and more dimensions later. We shall look at a very small part of the surface, as in Fig. 4.7. The change in the fuction due to taking small steps in both variables simultaneously (the most general one possible), is

$$f(x+\delta x, y+\delta y) - f(x,y) = \delta x\frac{f(x+\delta x,y) - f(x,y)}{\delta x} + \delta y\frac{f(x+\delta x,y+\delta y) - f(x+\delta x,y)}{\delta y} + \ldots, \qquad (4.2)$$

where, just as in one dimension, the thre dots denote terms of higher power in the small numbers δx and δy. The expression is not symmetric under the interchange of x and y, and we need to do one more step. The

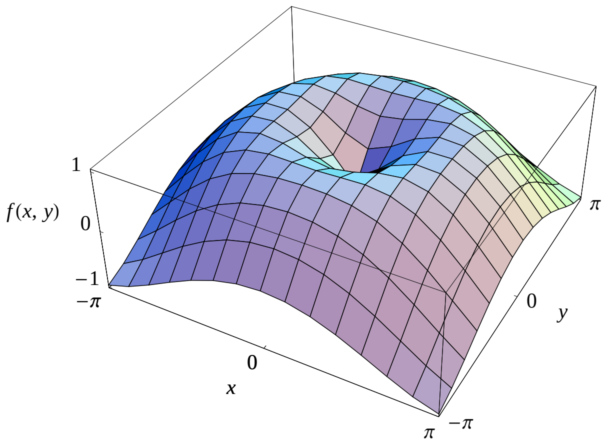


Figure 4.6: The surface $z = f(x, y) = \sin(\sqrt{x^2 + y^2})$.

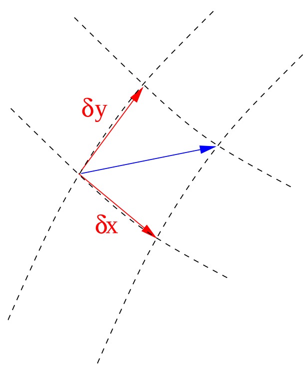


Figure 4.7: A small square δx by δy on the surface of a 2D function.

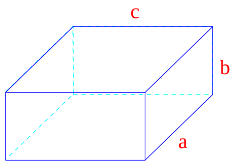


Figure 4.8: A cuboid $a \times b \times c$.

second term can be transformed back to refer to x rather than $x + \delta x$ by making an error proportional to δx. But that corresponds to a term $\delta x \delta y$ which is much smaller than the two terms already there if δx and δy are small. Thus

$$f(x+\delta x, y+\delta y) - f(x,y) = \delta x \frac{f(x+\delta x, y) - f(x,y)}{\delta x} + \delta y \frac{f(x, y+\delta y) - f(x,y)}{\delta y} + ..., \tag{4.3}$$

This show that a general change in the function can be referred back to a change in the individual variables, keeping the other fixed. In the limit of δx and δy going to zero this gives rise to the partial derivatives, denoted by a curly ∂. In mathematical notation

$$\frac{f(x+\delta x, y) - f(x,y)}{\delta x} \to \frac{\partial f}{\partial x} = \frac{d}{dx}\left(f(x,y)\right)_{y \text{ fixed}}, \tag{4.4}$$

$$\frac{f(x, y+\delta y) - f(x,y)}{\delta y} \to \frac{\partial f}{\partial y} = \frac{d}{dy}\left(f(x,y)\right)_{x \text{ fixed}}. \tag{4.5}$$

Example 4.16:

Given $u(x,y) = x^3 + x^2 y + xy + y^3$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Solution:

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 + 2xy + y + 0, \\ \frac{\partial u}{\partial y} &= 0 + x^2 + x + 3y^2. \end{aligned}$$

where the terms are the partial derivatives of each of the four terms in the function.

From $f(x+\delta x, y+\delta y) - f(x,y) \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$ we obtain that when both partial derivatives are zero we have an extremum (minimum or maximum or ...), where the fuction is "flat" in first approximation. We thus need to solve a simultaneous set of equations for such a thing to occur.

Example 4.17:

Calculate the minimum surface area for a cuboid of size $a \times b \times c$, Fig. 4.8, for fixed volume V.

Solution:

The volume V is simply abc. The surface is the area of the six rectangular sides, $S = 2ab+2ac+2bc$. The only problem is the constraint of constant volume. We can use that to eliminate one of the three variables from the problem, let us use c: $c = V/(ab)$. Thus

$$S = 2ab + \frac{2V}{b} + \frac{2V}{a}. \tag{4.6}$$

Now differentiate this with respect to a and b, and find

$$\begin{aligned} \frac{\partial S}{a} &= 2b - \frac{2V}{a^2}, \\ \frac{\partial S}{b} &= 2a - \frac{2V}{b^2}. \end{aligned} \tag{4.7}$$

These must both equal zero, and we get the equations

$$\begin{aligned} 2b &= \frac{2V}{a^2}, \\ 2a &= \frac{2V}{b^2}. \end{aligned} \tag{4.8}$$

Substitute the first equation into the l.h.s. of the second equation, and find

$$a = -Va^4, \tag{4.9}$$

which can be rewritten as $a(1 - Va^3) = 0$. Clearly the solution $a = 0$ is nonsensical (since b must be infinite), and we find

$$\boxed{a = b = c = V^{1/3},}$$

and the minimum surface is found for a cube.

4.4.1 Multiple partial derivatives

Multiple partial derivatives are defined straightofwardly as the partial derivative of the partial derivative,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \\ \frac{\partial^3 f}{\partial x^3} &= \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right). \end{aligned}$$

Slightly more complicated are the mixed ones,

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right). \end{aligned}$$

Even though this looks complicated, it can be shown that the order of differentiation actually doesn't matter!

Example 4.18:

Find all first and second derivatives of $f(x, y) = x \sin y + \cos(x - y)$.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \sin y - \sin(x - y), \\ \frac{\partial f}{\partial y} &= x \cos y + \sin(x - y), \\ \frac{\partial^2 f}{\partial x^2} &= 0 - \cos(x - y), \\ \frac{\partial^2 f}{\partial y^2} &= -x \sin y - \cos(x - y), \\ \frac{\partial^2 f}{\partial y \partial x} &= \cos y + \cos(x - y), \\ \frac{\partial^2 f}{\partial x \partial y} &= \cos y + \cos(x - y), \end{aligned}$$

where the last two terms have been calculated in the order indicated in the denominator, and we see the equality alluded to above.

4.5 Differentiation and curve sketching

4.6 Application of differentiation: Calculation of small errors

We know that if $y = f(x)$ then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \quad .$$

Provided that δx is small enough (but not infinitesimally small) $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$, so

$$\boxed{\delta y \approx \frac{dy}{dx}\delta x \quad .}$$

Example 4.19:

We can measure the volume of a sphere by measuring its radius r and then use the formula, $V = (4/3)\pi r^3$. Suppose we measure $r = 6.3 \pm 0.02$ m. Find the approximate error in V.

Solution:

If $r = 6.3$ m then $V = \frac{4}{3}\pi 6.30^3 = 1047.4$ m^3. The small error $\delta r = 0.02$ m will cause an error in V given by $\delta V \approx \frac{dV}{dr}\delta r = 4\pi r^2 \delta r = 4\pi\, 6.30^2\, 0.02 = 10.0$ m^3. Hence

$$\boxed{V = 1047.4 \pm 10.0 \text{ m}^3.}$$

Example 4.20:

We measure the height h of a tower at a distance d, by measuring d and the angle α with the horizontal. We then use the formula $\tan\alpha = (h/d)$.

Solution:

Find error in h due to an error $\delta\alpha$ in α assuming d to be known exactly. We solve for h, $h = d\tan\alpha$, $dh/d\alpha = d\sec^2\alpha$. Therefore

$$\boxed{\delta h \approx \frac{dh}{d\alpha}\delta\alpha = d\sec^2\alpha\delta\alpha \quad .}$$

Example 4.21:

Given the relation between current, voltage and resistance, $I = V/R$, with $V = 250$ V, $R = 50\ \Omega$, find the change in the current I if V changes by 1 V, and R by 1 Ω.

Solution:

We use the rule for small changes for partial derivatives,

$$\delta I \approx \frac{\partial I}{\partial V}\delta V + \frac{\partial I}{\partial R}\delta R\,.$$

We find

$$\begin{aligned} \frac{\partial I}{\partial V} &= \frac{1}{R}\,, \\ \frac{\partial I}{\partial R} &= \frac{-V}{R^2}\,. \end{aligned}$$

Using the numerical values, we find

$$\delta I = \frac{1}{50} \times 1 - \frac{250}{50^2} \times 1 = \frac{1}{50} - \frac{1}{10} = -\frac{2}{25} = -0.08 \text{ A}$$

Chapter 5

Integration

This chapter should contain partially things you know -essentially the basis of integration- and quite a few new things that build on that, extending your knowledge of integrals and integration.

5.1 What is integration?

There are two ways of thinking about integration, and they both have their uses

1. Inverse of differentiation
2. Area under a curve

The first is more useful for indefinite integrals (when there are no boundaries), and the second is only useful for definite ones (with upper and lower boundaries). We can use the first way to get results for the second interpretation, but not the other way around.

5.1.1 Inverse of differentiation

L&T, F.10.1

If $f(x) = \frac{dg(x)}{dx}$, then

$$\int f(x)dx = g(x) + c \quad ,$$

Example 5.1:

Integrate $4x^3$.

Solution:

$4x^3 = \frac{d(x^4)}{dx}$, so $\int 4x^3 dx = x^4 + c$

This type of integration is called an indefinite integral. We always get a constant of integration c for an indefinite integral.
Note: The result of $\int f(x)dx$ is another function of x. Many of the integrals in the formula book were obtained this way, Some examples:

$$\begin{aligned} \frac{d\sin x}{dx} &= \cos x & \int \cos x dx &= \sin x + c \\ \frac{d\ln x}{dx} &= \frac{1}{x} & \int \frac{1}{x} dx &= \ln x + c \\ \frac{d(e^{ax})}{dx} &= ae^{ax} & \int e^{ax} dx &= \frac{1}{a} e^{ax} + c \\ \frac{d(\sin^{-1} x)}{dx} &= \frac{1}{\sqrt{1-x^2}} & \int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x + c \end{aligned}$$

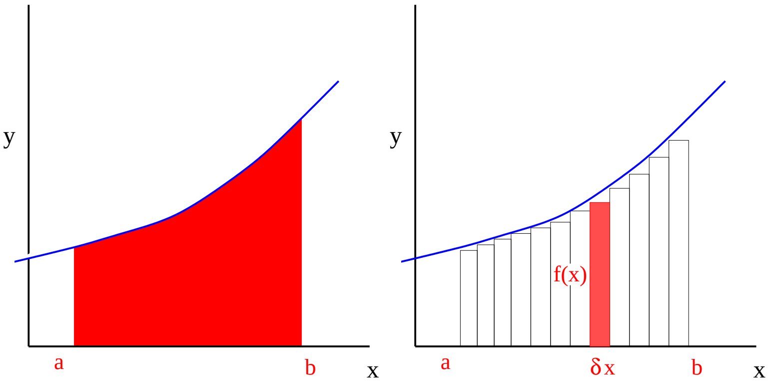


Figure 5.1: The area under a curve.

5.1.2 Area under a curve

L&T, F.10.22

Plot the curve $y = x$, as in Fig. 5.1. The shaded area under curve between $x = a$ and b equal a number A. We can calculate this as

$$A = \int_a^b x dx,$$

(This is called a definite integral.) This is defined as the sum from $x = a$ to $x = b$ of the area of all the small strips under the curve, in the limit that they become vanishingly thin.

If we know that $\int f(x)\,dx = g(x) + c$, than the resulting area is $A = g(b) - g(a)$.

Note: There is no constant of integration in a definite integral.

Note: The result is a number *not* a function.

Example 5.2:

Find area under curve $y = 2e^{3x}$ between $x = -1$ and $x = 1$.

Solution:

The area is given by the integral

$$\begin{aligned} A &= \int_{-1}^{1} 2e^{3x}\,dx \\ &= 2\int_{-1}^{1} e^{3x} dx \\ &= 2\left(e^{3x}/3\right)_{-1}^{1} \\ &= 2(e^3/3 - e^{-3}/3) \\ &\approx 13.3572 \quad . \end{aligned}$$

Final Remark: Some integrals can never be done in terms of known functions.

Example 5.3:

$\int e^{x^2}\,dx$, $\int_1^2 1/(x + \cos x)dx$. For these a numerical method will give results for a definite integral, e.g., a computer version of summing the area of the strips under a curve.

5.2 Strategy

Since there is no guaranteed method of doing integrals we proceed as follows

1. Draw up a list of as many as possible “standard integrals” that can be done. (This has already been done for you and is given in the formula book.)

2. When given a new integral you must try to rearrange into one of the standard types. This may involve some or all of the following

(a) directly rearrangement (rather trivial);

(b) substitution;

(c) integration by parts;

(d) special methods for particular types.

5.3 Integration by substitution

L&T, 15. (no exact match!)

This is the integral equivalent of the chain rule. If $z = f(x)$ and $x = g(t)$ then the chain rule says, $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}$. We can rearrange this "by multiplying by" dt to get,

$$\boxed{dz = \frac{dz}{dx}dx}$$

. (This can be proven from the rule for finite steps,

$$\frac{\delta z}{\delta t}\delta t = \frac{\delta z}{\delta x}\delta x,$$

which can be rearranged as

$$\delta z = \frac{\delta z}{deltax}\delta x.$$

In the limit that δx goes to zero, as it must in the integral, we find the required result). This is the basic formula we need to convert an integral with respect to a new variable z. It is true as a substitution rule inside the integral, not as a general equality.

5.3.1 Type 1

Replace some function of x by z.

Example 5.4:

Evaluate $I = \int_0^2 x\sin(x^2)\,dx$.

Solution:

Substitute $z = x^2$ (try this), then $dz = (dz/dx)dx = 2xdx$. We can only use this substitution if we can identify $2x\,dx$ as part of I. To that end write $I = (1/2)\int_0^2 \sin x^2\,2x\,dx$. We can now substitute for $x^2 = z$ and for $2xdx = dz$, and thus $I = \frac{1}{2}\int \sin z\,dz$, where the limits still need to be filled in. Since I is now an integral w.r.t. z, the limits must be starting and finishing values of z. At the start, where $x = 0$, $z = x^2 = 0$. At the finish $x = 2$, $z = 4$, so

$$\begin{aligned} I &= \frac{1}{2}\int_0^4 \sin z\,dz \\ &= \frac{1}{2}[-\cos z]_0^4 \\ &= \frac{1}{2}(-\cos 4 + 1) \\ &\approx 0.8268 \quad . \end{aligned}$$

Note: The integrand, (i.e., the object being integrated) changes from $x\sin(x^2)$ to $(1/2)\sin z$. Part of this change is due to the change from dx to dz.

Note: The integration limits change (for definite integrals only).

Example 5.5:

Calculate the indefinite integral $I = \int \frac{dx}{a^2 + x^2}$.

Solution:

Use substitution, and take $z = (x/a)$, $dz = (1/a)dx$, $x = az$.

$$\begin{aligned} I &= \int \frac{dx}{a^2 + x^2} a \frac{1}{a} dx \\ &= \int \frac{dx}{a^2 + a^2 z^2} a dz \\ &= \int \frac{a}{a^2(1 + z^2)} dz \\ &= \frac{1}{a} \int \frac{dz}{(1 + z^2)} \\ &= \frac{1}{a} \tan^{-1} z + c. \end{aligned}$$

Finally we must substitute back using $z = x/a$,

$$\boxed{I = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c.}$$

Several standard integrals can be generalised using this substitution (left as exercise).

Example 5.6:

Evaluate $I = \int \frac{1}{\sqrt{a^2 - x^2}} dx$.

Solution:

Using the substitution $x = az$ we find

$$I = \int \frac{dz}{\sqrt{1 - z^2}} = \sin^{-1} z + c = \sin^{-1}(x/a + c)$$

Thus

$$\boxed{\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}(x/a + c) \quad .}$$

5.3.2 Type 2

Replace x by a function of z. Sometimes, instead of putting

$$z = f(x) \quad , \tag{5.1}$$

e.g., $z = x^2$, we replace x directly by putting

$$x = g(z) \quad . \tag{5.2}$$

This is really same as using (5.1) since we can rearrange this equation, (i.e., solve for x) to get (5.2). However, we can work directly from (5.2) by calculating dx/dz. We then use the formula

$$dx = \frac{dx}{dz} dz \quad .$$

(Remember that we also must change limits on a definite integral!)

Example 5.7:

Evaluate $I = \int_0^4 \frac{1}{1+\sqrt{x}}\, dx$.

Solution:

Put $x = z^2$, $dx/dz = 2z$, $dx = 2zdz$. The limits change, $x = 0 \Rightarrow z = 0$, $x = 4 \Rightarrow z = 2$. We obtain

$$\begin{aligned} I &= \int_0^2 \frac{1}{1+\sqrt{z^2}} 2zdz = \int_0^2 \frac{2z}{(1+z)} dz \\ &= \int_0^2 \frac{2(z+1)-2}{(z+1)} dz = \int_0^2 \left(2 - \frac{2}{(z+1)}\right) dz \\ &= [2z - 2\ln(z+1)]_0^2 = (4 - 2\ln 3) - 0 = 4 - 2\ln 3 = 1.8028 \end{aligned}$$

5.4 Integration by Parts

L&T, 15,21-30

This is the integral equivalent to the differential of a product. Start with

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrate both sides with respect to x,

$$uv = \int u\frac{dv}{dx}\, dx + \int v\frac{du}{dx}\, dx \quad .$$

Now use $(dv/dx)dx = dv$ and $(du/dx)dx = du$. Rearrange the terms, and find

$$uv = \int udv + \int vdu.$$

This last equation is mainly used in the form

$$\boxed{\int u\, dv = uv - \int v\, du.}$$

Example 5.8:

Evaluate $I = \int xe^x\, dx$.

Solution:

Put $u = x$ and $e^x\, dx = dv$. u part: $u = x$, therefore $du/dx = 1$ and $du = dx$. v part: $e^x dx = dv$ therefore $dv/dx = e^x$ and $v = \int e^x\, du = e^x$ (constant of integration not needed here). Thus

$$\boxed{I = uv - \int v\, du = xe^x - \int e^x\, dx = xe^x - (e^x + c) \quad .}$$

Note that the x part of the original integrand (i.e., u) was differentiated, but the e^x part (i.e., dv/dx) was integrated. We obtained a new integral which was easier than the old one because (du/dx) was simpler than u but $\int v\, dx$ was no harder than v. It is a requirement that the resulting integral is no more complicated than the original!

Example 5.9:

Evaluate $I = \displaystyle\int x^2 \sin x dx$.

Solution:

Put $u = x^2$, $dv = \sin x\, dx$. $du/dx = 2x$, therefore $du = (2xdx)$. $dv/dx = \sin x$, therefore $v = \int \sin x\, dx = -\cos x$. We thus obtain

$$\begin{aligned} I &= uv - \int v\, du \\ &= x^2(-\cos x) - \int (-\cos x) 2x\, dx \\ &= -x^2 \cos x + 2 \int x \cos x\, dx\,. \end{aligned}$$

Now repeat this procedure: Put $u = x$, $\cos x\, dx = dv$. We find $du/dx = 1$, and therefore $du = dx$. Finally $dv/dx = \cos x$ and thus $v = \sin x$.

$$\begin{aligned} I &= -x^2 \cos x + 2 \left[x \sin x - \int \sin x\, dx \right] \\ &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + k. \end{aligned}$$

(We have put the constant of integration in at the end.)

Example 5.10:

Evaluate $I = \displaystyle\int \ln x\, dx$.

Solution:

Even though this does not look like integration by parts, we can use a trick! Use the fact that the derivative of the logarithm is much more manageable than the logarithm itself, and use a function v with derivative 1. Thus $u = \ln x$, $dv = 1\, dx$, $\frac{du}{dx} = \frac{1}{x}$, $\frac{dv}{dx} = 1$, $du = (1/x)dx$, $v = x$.

$$\begin{aligned} I &= uv - \int v\, du \\ &= x \ln x - \int x\, 1/x\, dx \\ &= x \ln x - \int 1\, dx \\ &= x \ln x - x + k \quad . \end{aligned}$$

Example 5.11:

Find $I = \displaystyle\int_0^{\pi/2} e^x \cos x\, dx$.

Solution:

Here we can integrate or differentiate e^x, and differentiate or integrate $\cos x$, since the integrals and derivatives of both functions are as simple as the original function. We choose $u = e^x$, therefore $du/dx = \cos x$ and $v = \sin x$.

$$\begin{aligned} I &= (uv)_0^{\pi/2} - \int_0^{\pi/2} v du \\ &= (e^x \sin x)_0^{\pi/2} - \int_0^{\pi/2} \sin x\ e^x\, dx \\ &= e^{\pi/2} - \int_0^{\pi/2} \sin x e^x\, dx. \end{aligned}$$

Now integrate by parts again.

Note: Initially we differentiated $u = e^x$, taking $\cos x$ as a derivative. We must use the *same* procedure again, and not switch u and v. I.e., we must put $u = e^x$ and $dv = \sin x dx$. Therefore $u = e^x$, $du/dx = e^x$, and thus $du = e^x dx$, $dv = \sin x dx$. It follows that $dv/dx = \sin x$, and so $v = -\cos x$.

$$\begin{aligned} I &= e^{\pi/2} - (-e^x \cos x)_0^{\pi/2} + \int_0^{\pi/2} -\cos x e^x\,dx \\ &= e^{\pi/2} - [0 - (-1)] - \int_0^{\pi/2} e^x \cos x\,dx \\ &= e^{\pi/2} - 1 - I \quad . \end{aligned}$$

Bring I to the left-hand side,

$$2I = e^{\pi/2} - 1.$$

and thus, finally,

$$\boxed{I = \frac{1}{2}\left(e^{\pi/2} - 1\right) \quad .}$$

5.5 Integrals of the form $I = \int(1)/(ax+b)\,dx$

The integral $I = \displaystyle\int(1)/(ax+b)\,dx$, can be done by substitution, $z = ax+b$, $dx = dz/a$, $I = \frac{1}{a}\int\frac{1}{z}\,dz = \frac{1}{a}(\ln z + C)$. Thus

$$\boxed{I = \int(1)/(ax+b)\,dx = \frac{1}{a}\,(\ln(ax+b) + C}$$

5.6 Integrals of the form $I = \int(px+q)/(x^2+ax+b)\,dx$

L&T, 15.31-43

We now study the integral $I = \int(px+q)/(x^2+ax+b)\,dx$, i.e., linear over quadratic, where the quadratic does not factorize.

Step 1 Calculate the differential of the denominator,

$$\frac{d}{dx}(x^2+ax+b) = 2x+a.$$

Use this to rearrange the numerator into form

$$\frac{p}{2}(2x+a) + (q - pa/2),$$

i.e., as a constant times the derivative of the denominator plus another constant. We can now split the integral,

$$I = \frac{p}{2}\int\frac{2x+a}{x^2+ax+b}\,dx + (q - pa/2)\int\frac{dx}{x^2+ax+b} \quad .$$

The first integral on the r.h.s. can be done using the substitution $z = x^2+ax+b$,

$$\int\frac{2x+a}{x^2+ax+b}\,dx = \int\frac{1}{z}\,dz = \ln z = \ln(x^2+ax+b) \quad .$$

Step 2 The second integral is more complicated, and we deal with

$$J = \int\frac{dx}{x^2+ax+b}$$

separately. The technique used is based on completing the square, $x^2 + ax + b = (x+c)^2 \pm d^2$, which after the substitution $z = x + c$ leads to a standard integral

$$\int \frac{dz}{z^2 \pm d^2}.$$

Depending on the sign we get either an inverse tangent or a ratio of logarithms,

$$\begin{aligned}\int \frac{1}{z^2+d^2}dz &= \frac{1}{d}\tan^{-1}(z/d) + c\,,\\ \int \frac{1}{z^2-d^2}dz &= \frac{1}{2d}\int\left(\frac{1}{z-d} - \frac{1}{z+d}\right)dz = \frac{1}{2d}\ln\left(\frac{z-d}{z+d}\right) + c\,.\end{aligned}$$

Example 5.12:

Evaluate $I = \displaystyle\int \frac{4x-1}{x^2+2x+3}\,dx$.

Solution:

Step 1 Differentiating the denominator gives $2x + 2$. Take apart into tow pieces, by rearranging numerator as $4x - 1 = 2(2x + 2) - 5$.

$$\begin{aligned} I &= \int \frac{2x+2}{x^2+2x+3}\,dx - 5\int\frac{dx}{x^2+2x+3}\\ &= \ln(x^2+2x+3) - 5\int\frac{dx}{x^2+2x-3}\end{aligned}$$

Now complete the square for the denominator, and find that

$$\begin{aligned} x^2+2x+3 &= (x+1)^2 + 2 = (x+1)^2 + \sqrt{2}^2\\ J = \int\frac{dx}{x^2+2x+3} &= \int \frac{dx}{(x+1)^2+\sqrt{2}^2}.\end{aligned}$$

Substitute $z = x + 1$, $dz = (dz/dx)dx = dx$,

$$\begin{aligned} J &= \int\frac{dz}{z^2+\sqrt{2}^2}\\ &= (1/\sqrt{2})\tan^{-1}(z/\sqrt{2}) + c \quad .\end{aligned}$$

Thus we find

$$\boxed{I = 2\ln(x^2+2x+3) - (5/\sqrt{2})\tan^{-1}((x+1)/\sqrt{2}) + c \quad .}$$

5.6.1 Completing the Square

L&T, 1.3.3.5

Completing the square is a simple idea that is surprisingly useful. First a definition:

A polynomial is a sum of powers of a variable x (say). The *degree* of this polynomial is its highest power.

Let us look at a few examples:

polynomial	degree	
(a) $x + 1$	1	Also called linear, since if we plot
(b) $4x$	1	the functions $y = x + 1$, $y = 4x$, etc.
(c) $ax + b$	1	we get a straight line
(d) $x^2 + 2x + 1$	2	
(e) $-7x^2 - 3$	2	(also known as quadratic)
(f) $ax^2 + bx + c$	2	
(g) $x^3/9 - \pi x$	3	cubic
(h) $12x^6 + 0.001$	6	

A polynomial of infinite degree is usually called an infinite power series.

Any polynomial of degree 2, i.e., a quadratic, can always be rearranged to have the form $a(x+b)^2+c$, as the square of a linear term plus a constant. Bringing a quadratic polynomial to this form is called *completing the square*.

5.6.2 Method

"Completing the square" is bringing a quadratic to the form $a(x+b)^2+c$.

In general, if two polynomials are equal, it means that the coefficient of each power of the variable are equal, since each power varies at a different rate with the variable. So in order to complete the square, we must equate coefficients of powers of x on both sides. We shall do this by example.

1. Complete the square in x^2+x+1:
   Put
   $$\begin{aligned} x^2+x+1 &= a(x+b)^2+c \\ &= ax^2+2abx+c+ab^2. \end{aligned}$$
   Now equate coefficients of x^2 on both sides. We find $1=a$, or $a=1$. Then compare the coefficients of x. We conclude $1=2ab$. Using $a=1$ we find $b=1/2$. Now equate the constant term, $1=ab^2+c=\frac{1}{4}+c$. We conclude that $c=3/4$.
   Collecting all the results we find
   $$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

2. Complete the square in $2x^2-x$.
   Solve $2x^2-x=a(x+b)^2+c$. We compare coefficients of
   x^2: $2=a$,
   x: $-1=2ab$, therefore $b=-1/4$,
   const: $0=ab^2+c$, therefore $c=-1/8$.
   Thus
   $$2x^2-x=2\left(x-\frac{1}{4}\right)^2-\frac{1}{8} \quad .$$

It is often useful to write the constant as

$$c=\begin{cases} d^2 & \text{(if } c \text{ is positive)} \\ -d^2 & \text{(if } c \text{ is negative)} \end{cases}$$

5.7 Integration of rational Functions

5.7.1 Partial fractions

L&T, 2.12.2.1

Before dealing with partial fractions, we need to define a rational function.

A rational function is one with the form $f(x)=P(x)/Q(x)$ $(Q(x)\neq 0)$, where $P(x)$ and $Q(x)$ are polynomials.

Partial fractions is a method of simplifying a rational function. For the present we shall only consider rational functions where the degree of the numerator is less than that of the denominator (*not* equal). If this is not true then we can convert it into this form–see later (integration section). First factorise the denominator $Q(x)$ into a mixture of linear and quadratic factors. This can always be done *without* using complex numbers (use linear factors only if possible). E.g.,

$$x^3-2x^2+x-12=(x-3)(x^2+x+4).$$

We can now simplify the rational function using partial fractions. We do this by means of examples as part of the revision.

Example 5.13:

Simplify $\frac{3x-1}{2x^2-x-1}$ using partial fractions.

Solution:

$$\frac{3x-1}{2x^2-x-1} = \frac{3x-1}{(x-1)(2x+1)}.$$

We have factorised the denominator). Now put

$$\frac{3x-1}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$$

(A, B constants). Multiply both sides by $(x-1)(2x+2)$, the denominator of the left-hand side. We find

$$3x - 1 = A(2x+1) + B(x-1). \tag{5.3}$$

Now compare coefficients on both sides. First x: $3 = 2A + B$, and for the constant term we find $-1 = A - B$. We solve these simultaneous linear equations, and find $A = \frac{2}{3}, B = \frac{5}{3}$. So

$$\boxed{\frac{3x-1}{(x-1)(2x+1)} = \frac{2}{3(x-1)} + \frac{5}{3(2x+1)}.}$$

Alternatively we can find A and B by choosing values for x. If we choose $x = 1$ then (5.3) becomes $2 = 3A + 0B$, and therefore $A = (2/3)$. If we choose $x = -(1/2)$ then it becomes $-(5/2) = 0A - (3/2)B$, and therefore $B = 5/3$, in agreement with our previous results.

Example 5.14:

Simplify $\frac{x+1}{x(x^2-4)}$ using partial fractions.

Solution:

$$\frac{x+1}{x(x^2-4)} = \frac{x+1}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2},$$

(left as an exercise, $A = -1/4$, $B = 3/8$, $C = -1/8$).

Example 5.15:

Simplify $\frac{x^2}{(x-1)(x-2)^3}$ using partial fractions.

Solution:

$$\frac{x^2}{(x-1)(x-2)^3} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3},$$

where we have one term for each power of the factor up to the maximum. Multiply by $(x-1)(x-2)^3$ and equate coefficients.

$$x^2 = A(x-2)^3 + B(x-1)(x-2)^2 + C(x-1)(x-2) + D(x-1).$$

Substitute $x = 2$: $4 = 0 + 0 + 0 + D$ so $D = 4$. $x = 1$: $1 = -A + 0 + 0 + 0$ so $A = -1$. Equate the coefficients of x^3: $0 = A + B + 0 + 0$, so $B = 1$, and the coefficients of constant term: $0 = -8A - 4B + 2C - D$, and thus $C = 0$.

$$\boxed{\frac{x^2}{(x-1)(x-2)^3} = -\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-2)^3}.}$$

Example 5.16:

Simplify $\frac{x+5}{x^3-1}$ using partial fractions.

Solution:

First factorise $Q(x)$, $x^3-1=(x-1)(x^2+x+1)$. We cannot factorise x^2+x+1 into their factors with real coefficients. Write

$$\frac{x+5}{x^3-1}=\frac{A}{x-1}+\frac{B+Cx}{x^2+x+1}.$$

Multiply with x^3-1,

$$x+5=A(x^2+x+1)+B+Cx(x-1),$$

substitute $x=1$: $3A=6$, or $A=2$. Equate coefficients of x^2: $A+C=0$, $C=-2$. Equate coefficients of the constant part: $A-B=5$, $B=-3$.

$$\boxed{\frac{x+5}{x^3-1}=\frac{2}{x-1}-\frac{3+2x}{x^2+x+1}.}$$

A rational function is a function of the form $f(x)=P(x)/Q(x)$ where P and Q are both polynomials.

Integration of such functions are dealt with according to the following procedure:

Step 1 If the degree of P is equal or greater that of Q then rearrange the numerator to get

$$P(x)=L(x)Q(x)+M(x) \tag{5.4}$$

where L and M are polynomials and M has lower degree than Q,

Example 5.17:

Bring $f(x)=\dfrac{2x^3+x^2+x+1}{x^3-x^2+2}$ to the form (5.4).

Solution:

Put $2x^3+x^2+x+1=2(x^3-x^2+2)+3x^2+x-3$ This corresponds to $L=2$ and $M=3x^2+x+3$. Thus $f(x)=\frac{LQ+M}{Q}=L+\frac{M}{Q}$. We can clearly integrate L directly (why?).

Step 2 We now have to integrate the new rational function $\frac{M}{Q}$ where M has lower degree than Q. This is dealt with by

1. factorising Q in linear and/or quadratic factors.
2. using the technique of partial fractions.

We now obtain integrals with one or more of the following types

(a) $\int \frac{1}{x+a}\,dx$: integrates to $\ln(x+a)$.

(b) $\int \frac{1}{(x+a)^2}\,dx$: integrates to $-\frac{1}{x+a}$

(c) $\int \frac{px+q}{x^2+ax+b}\,dx$: integrates see above (Sec. 5.6)

Example 5.18:

Integrate $\displaystyle\int(3x^2+x+3)/(x^3-x^2+2)\,dx.$

Solution:

This integrand can be rewritten as

$$\frac{3x^2+x+3}{x^3-x^2+2} = \frac{3x^2+x+3}{(x+1)(x^2-2x+2)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-2x+2} \quad .$$

To find A, B, C, we need to solve

$$3x^2+x-3=A(x^2-2x+2)+(Bx+C)(x+1) \quad .$$

We can get one of the values for almost free, using $x=-1$: $5A=-1$, or $A=-1/5$. We solve for the rest by equating the coefficients of identical powers of x,

x^2: $3 = A + B$, therefore $B = 16/5$,
constant: $-3 = 2A + C$, so that $C = -13/5$.

We have reexpressed the integral as

$$\int \frac{3x^2 + x + 3}{x^3 - x^2 + 2}\,dx = -\frac{1}{5}\int \frac{1}{x+1}\,dx + \frac{1}{5}\int \frac{16x - 13}{x^2 - 2x + 2}\,dx.$$

The first term $(1/(x+1))$ is easy to integrate and gives $\ln(x+1)$. Let us therefore concentrate on the second term

$$\begin{aligned}\int \frac{16x - 13}{x^2 - 2x + 2}\,dx &= \int \frac{8(2x-2)+5}{x^2 - 2x + 2}\,dx \\ &= 8\int \frac{(x^2 - 2x + 2)'}{x^2 - 2x + 2}\,dx + \int \frac{5}{x^2 - 2x + 2}\,dx \\ &= 8\ln(x^2 - 2x + 2) + \int \frac{5}{x^2 - 2x + 2}\,dx.\end{aligned}$$

Here we have used the fact that the differential of the denominator is $2x-2$. The remaining integral is treated by completing the square,

$$x^2 - 2x + 2 = (x-1)^2 + 1,$$

which allows us to write

$$\int \frac{5}{x^2 - 2x + 2}\,dx = 5\tan^{-1}(x-1)$$

Using the two previous examples we conclude that

$$\boxed{\int \frac{2x^3 + x^2 + x + 1}{(x^3 - x^2 + 2)}\,dx = 2x - \frac{1}{5}\ln(x+1) + \frac{8}{5}\ln(x^2 - 2x + 2) - \frac{3}{5}\tan^{-1}(x-1) \quad .}$$

5.8 Integrals with square roots in denominator

L&T, 16. (some overlap).

We shall consider only one type

$$\int \frac{1}{\sqrt{a + bx - x^2}}\,dx \quad .$$

The coefficient of x^2 must be negative, if it is positive we need a different approach which involves hyperbolic functions (not discussed here). The method is as follows

1. Complete the square, $a + bx - x^2 = d^2 - (x+c)^2$, with $c = -b/2$ and $d^2 = a + b^2/4$.
2. Substitute $z = x - c$, which gives us the derivative of the arcsin.

Example 5.19:

Calculate $I = \displaystyle\int \frac{dx}{\sqrt{3 + 4x - x^2}}$.

Solution:

Complete the square: $3 + 4x - x^2 = d^2 - (x+c)^2$. Equate the coefficients of each power. x^2: $-1 = -1$, contains no unknowns. x: $4 = -2c$ (therefore $c = -2$). The constant term gives $3 = d^2 - c^2 = d^2 - 4$, and thus $d^2 = 7$, $d = \sqrt{7}$, and

$$I = \int \frac{dx}{\sqrt{7 - (x-2)^2}}.$$

We substitute $z = x - 2$, $dz = dz/dx\, dx = dx$, which leads to

$$\boxed{I = \int \frac{dz}{\sqrt{\sqrt{7}^2 - z^2}} = \sin^{-1} \frac{z}{\sqrt{7}} + k = \sin^{-1}\left(\frac{x-2}{\sqrt{7}}\right) + k.}$$

(The integral is a standard integral and can be found in the tables, but is easily checked by using the chain rule and

$$\frac{d}{dy} \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}. \quad)$$

Chapter 6

Applications of Integration

6.1 Finding areas

L&T, 18.1-18

We have already discussed how an integral corresponds to an area.

Example 6.1:

Evaluate the area A under $y = x^2$ from $x = 1$ to $x = 3$.

Solution:

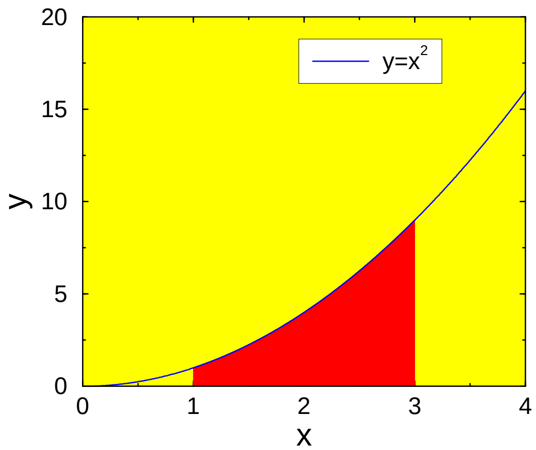
$A = \int_1^3 x^2\,dx$ which is $27/3 - 1/3 = 26/3$, see Fig. 6.1.

6.1.1 Area between two curves

Example 6.2:

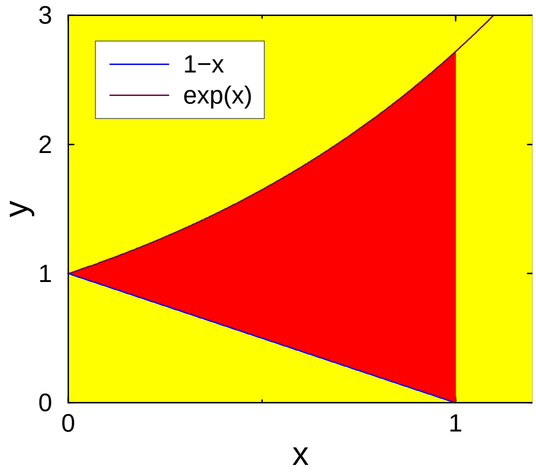
Find the area A of the region bounded by $y = e^x$ and $y = 1 - x$, for x ranging from 0 to 1, see Fig. 6.2.

Solution:



[htb]

Figure 6.1: The surface below x^2 between 1 and 3.



[htb]

Figure 6.2: The area between $1 - x$ and e^x for x between 0 and 1.

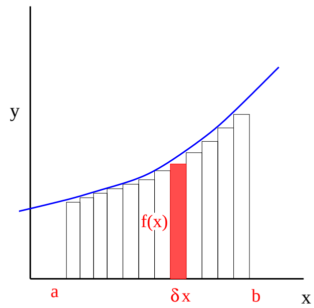


Figure 6.3: Integration as the sum of area of small strips.

From the graph we see that e^x is above $1 - x$, so that

$$\begin{aligned} A &= (\text{area below } y = e^x) - (\text{area below } y = 1 - x) \\ &= \int_0^1 e^x \, dx - \int_0^1 (1 - x) \, dx \\ &= \int_0^1 (e^x - 1 + x) \, dx \\ &= \left(e^x - x + \frac{x^2}{2} \right)_0^1 \\ &= \left(e - 1 + \frac{1}{2} \right) - 1 \\ &= e - 2 + \frac{1}{2} \\ &\approx 1.2183 \quad . \end{aligned}$$

Here we have made the optional choice to combine the two integrands before evaluation of the integral.

6.1.2 Basic Derivation of Area Formula

L&T, 18.1-18

To find area beneath the curve $y = f(x)$ between $x = a$ and $x = b$, we divide the area into strips as shown in Fig. 6.3. Let the thickness of strip at x be δx. The height at x is $f(x)$, and therefore the area of the strip

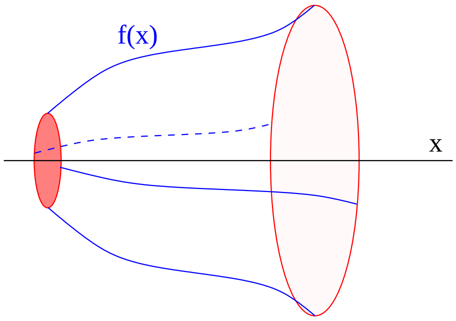


Figure 6.4: A surface of revolution.

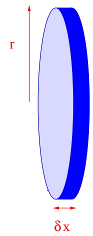


Figure 6.5: The volume of a small disc.

is $\delta A \approx f(x)\delta x$. Now sum up all strips from a to b. The areas is

$$A \approx \sum_a^b f(x)\delta x.$$

In the limit that δx becomes infinitesimal (i.e., approaches zero), we replace δx by dx, the $\sum_a^b$ by $\int_a^b$ and so

$$A = \int_a^b f(x)\,dx. \tag{6.2}$$

6.2 Volumes of Revolution

L&T, 19.1-11

If we take area under the curve $y = f(x)$ between $x = a$ and $x = b$, as above and then rotate it around the x axis through 360° we sweep out a volume called a volume of revolution V.

This situation is shown in Fig. 6.4. Clearly V has an axis of symmetry, i.e., the x axis. Many volumes that occur in practice have such an axis. We can use integration to find the volume.

Again divide the area into strips of width δx. Since the height is $f(x)$, when we rotate the strip we get a disc of radius $r = f(x)$, see Fig. 6.5. The area of this disc is $\pi r^2 = \pi f(x)^2$, and the volume of the disc is $\delta V = \pi r^2 \delta x$. The total volume is again a sum,

$$V = \sum_a^b \pi r^2 \delta x = \pi \sum_a^b f(x)^2 \delta x.$$

Now take limit where δx becomes infinitesimal, and thus

$$V = \pi \int_a^b f(x)^2\,dx.$$

This is the formula for the volume of a solid of revolution.

Example 6.3:

Find the volume formed when the curve $y = 1/x$, between $x = 1$ and $x = 2$ is rotated around the x axis, see Fig. 6.6

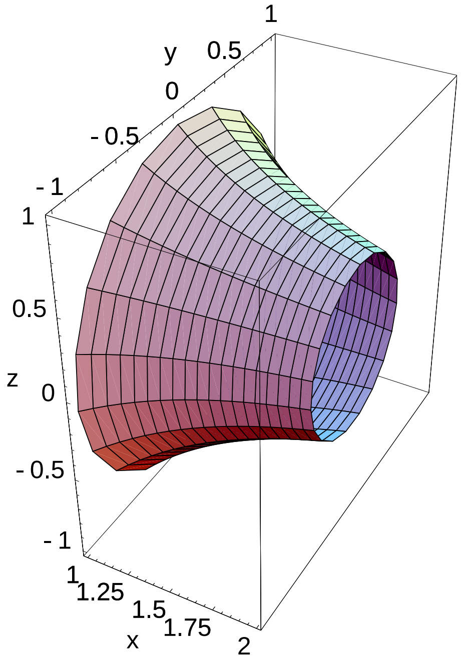


Figure 6.6: The surface of revolution for $y = (1/x)$, $1 < x < 2$.

Solution:

$$\begin{aligned} V &= \pi \int_1^2 (1/x)^2\, dx \\ &= \pi\,(-1/x)_1^2 \\ &= \pi(-(1/2) - (-1)) \\ &= \pi/2 \quad . \end{aligned}$$

Example 6.4:

Find the volume formed when equilateral triangle with corners at $O = (0, 0)$, $A = (1, \sqrt{3})$, $B = (2, 0)$ is rotated around the x axis, see Fig. 6.7.

Solution:

Along OA the curve is $y = \sqrt{3}x$, along AB the curve is $y = 2\sqrt{3} - \sqrt{3}x$. Thus

$$\begin{aligned} V &= \pi \int_0^1 (\sqrt{3}x)^2\, dx + \pi \int_1^2 (2\sqrt{3} - \sqrt{3}x)^2\, dx \\ &= 3\pi \left(x^3/3\right)_0^1 + 3\pi \left(-(2-x)^3/3\right)_1^2 \\ &= \pi + \pi(0 + 1) \\ &= 2\pi. \end{aligned}$$

6.3 Centroids (First moment of area)

L&T, 19.12-22

6.3.1 First moment of the area about the y axis

Again consider curve $y = f(x)$ from a to b, divided into strips of thickness δx. The area of the strip is given by $(\delta A \approx f((x))\delta x)$. The total area is given by the sum,

$$A \approx \sum_a^b \delta A = \sum_a^b f(x)\delta x \to \int_a^b f(x)\, dx.$$

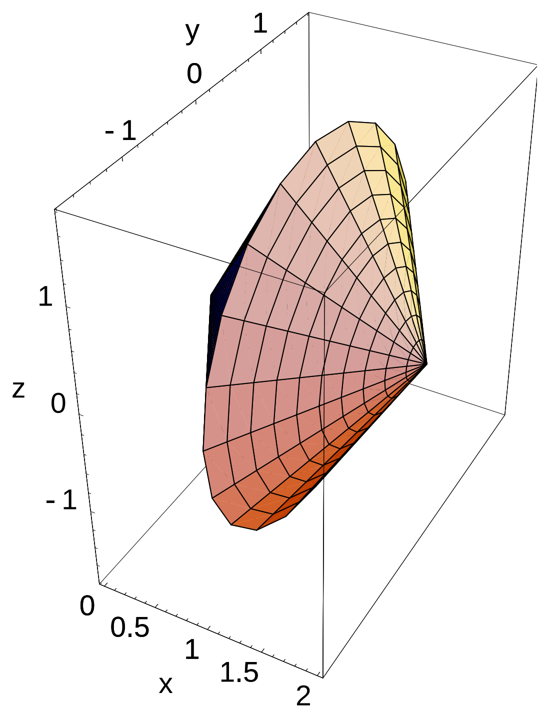


Figure 6.7:

If the strip is very thin then all of it is approximately at a distance x from y axis. If we now add up NOT δA but instead δA times x, i.e., δA "weighted" by x, we get the first moment of the area about the x axis,

$$M_x \approx \sum_a^b x\delta A = \sum_a^b xf(x)\delta x \to \int_a^b xf((x))\,dx \quad .$$

This is usually called M_x, even though it is the first moment around the y axis.

Example 6.5:

Find the first moment of area under $y = 1 + x + x^2$ from $x = 0$ to $x = 2$ about the y axis.

Solution:

$$\begin{aligned} M_x &= \int_0^2 x(1 + x^2 + x^3)\,dx \\ &= \int_0^2 (x + x^2 + x^3)\,dx \\ &= \left(x^2/2 + x^3/3 + x^4/4\right)_0^2 \\ &= 2 + 8/3 + 4 \\ &= 26/3 \quad . \end{aligned}$$

Example 6.6:

Find the first moment of the area under $y = e^{-x}$ from $x = 0$ to $x = 1$ about the y axis.

Solution:

$$M_x = \int_0^1 xe^{-x}\,dx.$$

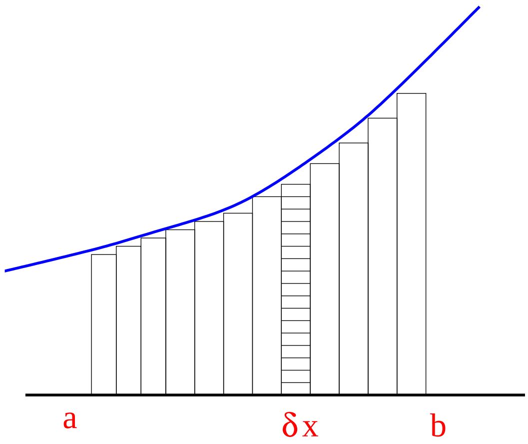


Figure 6.8: Subdividing the strips of width δx in ones of height δy.

Integrate by parts: $u = x$, $du/dx = 1$, $du = dx$, $dv = e^{-x}dx$. Therefore $dv/dx = e^{-x}$, and thus $v = -e^{-x}$,

$$\begin{aligned} M_x &= \left(xe^{-x}\right)_0^1 - \int_0^1 (-e^{-x})\, dx \\ &= -\frac{1}{e} - 0 + \int_0^1 e^{-x}\, dx \\ &= -\frac{1}{e} + (-e^{-x})_0^1 \\ &= -\frac{1}{e} + (-\frac{1}{e} + 1) \\ &= 1 - \frac{2}{e} = 0.2642 \quad . \end{aligned}$$

6.3.2 First Moment of the area about the x axis

Now consider the same strip of thickness δx. On this strip y goes from 0 to $f(x)$. Divide strip into segments of length δy as shown in Fig. 6.8. The area of such a segment is $\delta y \delta x$. The total area of strip is $\delta A \approx \sum_{y=0}^{f(x)} \delta y \delta x$. In the limit that δy becomes infinitesimal we get

$$\begin{aligned} \delta A \quad & \int_{y=0}^{f(x)} dy \delta x \\ &= (y)_0^{f(x)} \delta x \\ &= f(x) \delta x, \end{aligned}$$

as before. Now instead of summing segments we can weight each of them by the value of y to get

$$\begin{aligned} \delta M_y &= \sum_{y=0}^{f(x)} y \delta y \delta x \\ &= (\int_0^{f(x)} y\, dy) \delta x \\ &= (\frac{y^2}{2})_0^{f(x)} \delta x \\ &= \frac{1}{2} f(x)^2 \delta x \end{aligned}$$

To find M_y we have to add the contributions of all strips

$$\begin{aligned} M_y &= \sum_a^b \delta M_y \\ &= \sum_a^b \frac{1}{2} f(x)^2 \delta x \\ &= \frac{1}{2}\int_a^b f(x)^2\, dx \end{aligned}$$

This is the formula for the first moment of the area about the x axis (This integral is same as that for the volume of revolution except for the factor $\frac{1}{2}$ outside the integral rather than π).

Example 6.7:

Find M_y for area under curve $y = 1+x+x^2$ from $x = 0$ to $x = 2$ (same area as in example xxxx(1))

Solution:

$$\begin{aligned} f(x) &= 1+x+x^2 \quad , \\ f(x)^2 &= (1+x+x^2)^2 \\ &= 1+2x+3x^2+2x^3+x^4 \quad . \end{aligned}$$

Therefore

$$\begin{aligned} M_y &= \frac{1}{2}\int_0^2 (1+2x+3x^2+2x^3+x^4)\, dx \\ &= \frac{1}{2}\left(x+x^2+x^3+\frac{x^4}{2}+\frac{x^5}{5}\right)_0^2 \\ &= \frac{1}{2}\left(2+4+8+8+\frac{32}{5}\right) \\ &= 11+\frac{16}{5} \\ &= \frac{71}{5} \\ &= 14.2 \quad . \end{aligned}$$

6.3.3 Centroid of a plane area

For any plane shape with area A, the centroid is a point with coordinates (x_C, y_C) given by $x_C = 1/AM_x$, $y_C = 1/AM_y$, where M_x is first moment of area about the y axis, and M_y is first moment of area about the x axis

Example 6.8:

Find the centroid of the area under $y = 1+x+x^2$ from $x = 0$ to $x = 2$ using the previous two examples.

Solution:

We know that $M_x = 26/3$ and $M_y = 71/5$, and we just need to determine A,

$$\begin{aligned} A &= \int_0^2 (1+x+x^2)\, dx \\ &= (x+x^2/2+x^3/3)_0^2 \\ &= 2+2+8/3 \\ &= 20/3 \quad . \end{aligned}$$

Therefore

$$\begin{aligned} x_C &= \frac{M_x}{A} = \frac{3}{20}\frac{26}{3} = \frac{26}{20} = 1.3 \quad , \\ y_C &= \frac{M_y}{A} = \frac{3}{20}\frac{71}{5} = \frac{213}{100} = 2.13 \quad . \end{aligned}$$

6.3.4 Meaning of the centroid

If we have thin plate with constant thickness then the centroid is the position of centre of mass (C of M). The C of M is the point at which all mass can be regarded as acting. Let mass per unit area be ρ: This will be constant if the thickness is constant (and material is of uniform composition). The total mass $m = A\rho$ where A is area. Turning effect about y axis of mass m at (x, y) would be $mx = A\rho x$. A strip of thickness δx, height $f(x)$ has area $f(x)\delta x$. Mass would be $\rho f(x)\delta x$. Total turning effect is $\sum_a^b x\rho f(x)\delta x \to \int_a^b xf(x)dx = \rho M_x$, therefore $A\rho x = \rho M_x$, therefore $x_C = 1/AM_x$.

6.4 Second Moment of Area

The first moment of area (about the y axis) was

$$M_x \approx \sum_a^b x\delta A = \sum_a^b xf(x)dx \to \int_a^b xf(x)\,dx.$$

Similarly second moment is same but with x^2 instead of x,

$$\begin{aligned} \delta x &= \sum_a^b x^2 \quad , \\ \delta A &= \sum_a^b x^2 f(x)dx \to \int_a^b x^2 f(x)\,dx \quad . \end{aligned}$$

Example 6.9:

Find the second moment of area under $y = 1 + x + x^2$ about the y axis from $x = 0$ to $x = 2$.

Solution:

$$\begin{aligned} \delta x &= \int_0^2 x^2(1 + x^2 + x^3)\,dx \\ &= \int_0^2 (x^2 + x^3 + x^4)\,dx \\ &= \left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}\right)_0^2 \\ &= \frac{8}{3} + \frac{16}{4} + \frac{32}{5} \\ &= \frac{40 + 60 + 96}{15} \\ &= \frac{196}{15} = 13\frac{1}{15}. \end{aligned}$$

Note: To find second moment about x axis is more complicated:

$$\delta y = \int_a^b \frac{1}{3} f(x)^3\,dx.$$

This will not be done here.

Note: Recall that first moments are used in calculating centroids which are related to centres of mass. Second moments are used in calculating moments of inertia of flat planes.

Chapter 7

Differential Equations

7.1 introduction

A differential equation (DE) is any equation with a differential in it

Examples:

(a) $\frac{dy}{dx} = 1$,

(b) $\frac{dy}{dx} = x^2$,

(c) $\frac{dy}{dx} = y$,

(d) $x\frac{dy}{dx} + 2y = \cos x$,

(e) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$.

Differential equations occur in many models of real-world situations. One particular examples when we consider rates of change, e.g.,

(f) the concentration in C a chemical reaction $\frac{dC}{dt} = a - kC$,

(g) (explosive) population growth $\frac{dN}{dt} = \alpha N$,

(h) simple harmonic motion $\frac{d^2x}{dt^2} = -\omega^2 x$,

(i) motion under the influence of the earth gravitional field, $m\frac{d^2y}{dt^2} = -mg$.

We would really like to classify such equations by their *order*.

The order of a DE is the highest derivative contained in it.

Thus (a), (b), (c), (d), (f), (g) are first order, and (c), (h), (i) are second order. **In this course we only consider first order DEs**.

Solving DEs is sometimes called integrating them, since for the simplest types this is exactly what we do. Just as for integration we draw up a list of standard types that we know how to do.

Most solutions of DEs contain constants. These are just like constants of integration, and arise from the fact that the derivatives of these constants is 0. We always get as many arbitrary constants as the order of the equation. The general solution will include these arbitrary constants. If we have extra information apart from the DE itself we can find the arbitrary constants. This extra information is sometimes called "initial conditions" or "boundary conditions". Once the arbitrary constants are known we have the *actual solution*.

We shall study several types of DEs to facilitate solution, but let's first look at two simple examples.

Example 7.1:

Solve the DE

$$\frac{dy}{dx} = x^2 - 2,$$

given that $y = 1$ for $x = 0$.

Solution:

Integrate both sides of the equation,

$$y = \int \frac{dy}{dx} dx = \int (x^2 - 2) dx = \frac{1}{3}x^3 - 2x + k.$$

At $x = 0$, $y = 1$, which implies $k = 1$. Thus

$$\boxed{y(x) = \frac{x^3}{3} - 2x + 1.}$$

Example 7.2:

Find the general solution of $\cos x \, dy/dx + 2\sin x = 0$.

Solution:

Rearrange as

$$\begin{aligned} \frac{dy}{dx} &= -2\frac{\sin x}{\cos x} \\ &= -2\tan x \quad . \end{aligned}$$

Integrate

$$\boxed{y = -2\ln(\sec x) + k.}$$

7.2 Some special types of DE

7.2.1 Separable type

Equations of the form

$$dy/dx = f(x)g(y)$$

are called separable. They are dealt with in the following way: Divide both sides by $g(y)$, and integrate both sides with respect to x,

$$\begin{aligned} \int \frac{1}{g(y)} \frac{dy}{dx} dx &= \int f(x) \, dx, \\ \int \frac{1}{g(y)} dy &= \int f(x) dx \quad . \end{aligned}$$

Now do both integrals.

Example 7.3:

Solve the DE

$$\frac{dy}{dx} = 2xy^2$$

, given that $y = 1/2$ when $x = 0$.

Solution:

Divide by y^2, and obtain

$$\frac{1}{y^2} \frac{dy}{dx} = 2x.$$

Now integrate both sides with respect to x

$$\begin{aligned} \int \frac{1}{y^2} \frac{dy}{dx} dx &= \int 2x \, dx \quad , \\ \int \frac{1}{y^2} dy &= x^2 + k \quad , \\ -\frac{1}{y} &= x^2 + k \quad , \\ y &= -\frac{1}{x^2 + k} \quad . \end{aligned}$$

This is the general solution, but we know that at $x = 0$, $y = 1/2$. Substituting this we find that $1/2 = -1/k$, therefore $k = -2$ and

$$\boxed{y = -\frac{1}{x^2-2} = \frac{1}{2-x^2}.}$$

Example 7.4:

Find the general solution of

$$2y(x+1)\frac{dy}{dx} = 4 + y^2.$$

Solution:

Rearrange as

$$\frac{dy}{dx} = \frac{4+y^2}{2y(x+1)}.$$

So here $f(x) = 1/(x+1)$, $g(y) = \frac{4+y^2}{2y}$. Divide by $g(y)$,

$$\frac{2y}{4+y^2}\frac{dy}{dx} = \frac{1}{x+1} \quad .$$

Integrate both sides with respect to x

$$\begin{aligned}
\int \frac{2y}{4+y^2}\frac{dy}{dx}\,dx &= \int \frac{1}{x+1}\,dx \quad ,\\
\int \frac{2y}{4+y^2}dy &= \ln(x+1) + k \quad ,\\
\ln(4+y^2) &= \ln(x+1) + k \quad .
\end{aligned}$$

We write $k = \ln A$, with A also arbitrary, but positive. We find

$$\ln(4+y^2) = \ln(x+1) + \ln A = \ln(A(x+1)),$$

Thus $4 + y^2 = A(x+1)$, or isolating y,

$$\boxed{y = \pm\sqrt{A(x+1)-4}.}$$

Example 7.5:

$N(t)$ satisfies the DE

$$\frac{dN}{dt} = \alpha N.$$

Given that $N = 10$ at $t = 0$ find N at $t = 3$.

Solution:

Here $f(t) = \alpha$, i.e., a constant, and $g(N) = N$, so

$$\begin{aligned}
1/N dN/dt &= \alpha \quad ,\\
\int \frac{1}{N}\frac{dN}{dt}\,dt &= \int \alpha\,dt \quad ,\\
\int \frac{1}{N}\,dN &= \alpha t + k \quad ,\\
\ln N &= \alpha t + k \quad ,\\
N &= e^k e^{\alpha t} \quad ,\\
N &= Ae^{\alpha t} \quad .
\end{aligned}$$

Since at $t = 0$, $N = 10$, we have $A = 10$, and

$$\boxed{N = 10e^{\alpha t} \quad .}$$

At $t = 3$, $N = 10e^{3\alpha}$.

7.2.2 linear type

These have form,

$$\frac{dy}{dx} + p(x)y = q(x) \tag{7.1}$$

Method as follows

Step 1 Find indefinite integral of $p(x)$ and call this $s(x)$ (no constant of integration needed),

$$s(x) = \int p(x)\, dx \quad ,$$

and thus $ds/dx = p$.

Step 2 Multiply both sides of (7.1) by $e^{s(x)}$,

$$e^s \frac{dy}{dx} + e^s py = e^s q \quad .$$

Since $p = ds/dx$ we have

$$e^s \frac{dy}{dx} + e^s \frac{ds}{dx} y = e^s q \quad . \tag{7.2}$$

Step 3 Note that

$$\frac{d}{dx}(e^s) = \frac{d}{ds}(e^s)\frac{ds}{dx} = e^s \frac{ds}{dx} \quad .$$

Note also that

$$\frac{d}{dx}(ye^s) = \frac{dy}{dx}e^s + y\frac{d}{dx}(e^s) = (e^s)\frac{dy}{dx} + y(e^s)\frac{ds}{dx} \quad .$$

This is exactly the l.h.s. of (7.2). Rewrite eq. (7.2) as

$$\frac{d}{dx}(ye^s) = qe^s \quad . \tag{7.3}$$

Step 4 Integrate both sides with respect to x,

$$ye^s = \int qe^s\, dx + k \quad .$$

Hence

$$\boxed{y = e^{-s}\left[\int qe^s\, dx + k\right] \quad .}$$

N.B. Remember the method *not the final formula*!

Example 7.6:

Find the general solution of

$$\frac{dy}{dx} + (\tan x)y = 3\cos x \quad . \tag{7.4}$$

Solution:

Here $p = \tan x$ so $s = \int \tan x\, dx = \ln(\sec x)$ (no constant of integration needed here), $e^s = e^{\ln(\sec x)} = \sec x$. Multiply both sides of (7.4) by $e^s = \sec x$:

$$\sec x \frac{dy}{dx} + \sec(x)\tan(x)y = 3\cos(x)\sec(x) \quad .$$

The l.h.s. is the differential of $e^s y$ so we find

$$\frac{d\sec(x)y}{dx} = 3 \quad .$$

Integrate this and find

$$(\sec x)y = 3x + k \quad .$$

Thus, finally,

$$\boxed{y = (3x + k)\cos x \quad .}$$

Example 7.7:

Solve the DE

$$x\frac{dy}{dx} + 2y = 4x, \tag{7.5}$$

given that $y = 0$ when $x = 1$.

Solution:

Rearrange (7.5) ,

$$\frac{dy}{dx} + \frac{2}{x}y = 4 \quad , \tag{7.6}$$

which is of linear form with $p = \frac{2}{x}$. We find

$$s \quad = \quad \int p\,dx = \int \frac{2}{x}\,dx = 2\ln x = \ln(x^2) \quad ,$$

and $e^s = e^{\ln(x^2)} = x^2$. Multiply (7.6) by $e^s = x^2$, and find

$$x^2\frac{dy}{dx} + 2xy = 4x^2 \quad .$$

the l.h.s. is differential of $e^s y = x^2 y$. Integrate this and find

$$\begin{aligned} x^2 y &= 4x^3/3 + k \quad \text{, or} \\ y &= 4x/3 + k/x^2 \quad . \end{aligned}$$

This is the general solution. We know that when $x = 1$ then $y = 0$ so

$$0 = 4/3 + k/1$$

. Therefore $k = -4/3$ and

$$\boxed{y = \frac{4}{3}(x - \frac{1}{x^2}) \quad .}$$

7.2.3 Homogeneous Type

We first need to define a function of two variables:

> If $f(x, y)$ is a function of 2 variables, it delivers a number on specification of x and y.

Examples:
$x + y$, $y\cos(\pi x)$, $\frac{\ln y}{x^2+y^2}$.
If $x = 1$ and $y = 2$ in the above we get 3, -2, $\frac{1}{5}\ln 2$.
Now we can define a homogeneous function:

> A homogeneous function of 2 variables is one where we have a sum of terms all of which have the same total power (called degree).

Examples

function	degree
$x^2 + xy + y^2$	2
$x + 2y$	1
$\frac{x^2}{y} + \frac{y^2}{x}$	1
$1 + \frac{x}{y} + \frac{x^2}{y^2}$	0
xy	2
$x^2 + y$	**not** homogeneous
$x + y + 1$	**not** homogeneous

There is a simple test to see if $f(x, y)$ is homogeneous. Replace x by λx and y by λy to get $f(\lambda x, \lambda y)$. If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then f is homogeneous with degree n.

Example 7.8:

(a) $f(x,y) = x + 2y$:

$$f(\lambda x, \lambda y) = \lambda x + 2\lambda y = \lambda(x+2y) \quad ,$$

and f is homogeneous with degree 1.

(b) $f(x,y) = 1 + \frac{x}{y} + \frac{x^2}{y^2}$:

$$\begin{aligned} f(\lambda x, \lambda y) &= 1 + \frac{\lambda x}{\lambda y} + \frac{\lambda^2 x^2}{\lambda^2 y^2} \\ &= 1 + \frac{x}{y} + \frac{x^2}{y^2} \\ &= 1 f(x,y) \\ &= \lambda^0 f(x,y) \quad , \end{aligned}$$

which is therefore homogeneous of degree 0.

(c) $f(x,y) = \cos\left(\frac{x}{y}\right)$.

$$f(\lambda x, \lambda y) = \cos\left(\frac{\lambda x}{\lambda y}\right) = \cos\left(\frac{x}{y}\right)$$

which is therefore homogeneous of degree 0.

A homogeneous DE is one of type $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, with g and f both homogeneous and of the same degree.

Homogeneous DEs can be made separable by the substitution $y = xv$. We shall demonstrate this by means of examples:

Example 7.9:

Find general solution of

$$\frac{dy}{dx} = \frac{x+2y}{2x-y} \quad .$$

Solution:

Put $y = xv(x)$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$, so

$$v + x\frac{dv}{dx} = \frac{x+2xv}{2x-xv} = \frac{1+2v}{2-v} \quad .$$

therefore

$$\begin{aligned} x\frac{dv}{dx} &= \frac{1+2v}{2-v} - v = \frac{1+v^2}{2-v} \quad , \\ \frac{dv}{dx} &= \frac{1}{x}\left(\frac{1+v^2}{2-v}\right) \end{aligned}$$

which is separable. This can be solved in the standard way,

$$\begin{aligned} \left(\frac{2-v}{1+v^2}\right)\frac{dv}{dx} &= \frac{1}{x} \quad , \\ \int \left(\frac{2-v}{1+v^2}\right)\frac{dv}{dx}\,dx &= \int \frac{1}{x}\,dx \quad , \\ \int \frac{2-v}{(1+v^2)}\,dv &= \ln x + k \quad , \\ \int \frac{2}{(1+v^2)}\,dv - \frac{1}{2}\int \frac{2v}{(1+v^2)}\,dv &= \ln x + k \quad , \\ 2\tan^{-1} v - \frac{1}{2}\ln(1+v^2) &= \ln x + k \quad . \end{aligned}$$

And we conclude that

$$\boxed{2\tan^{-1}\left(\frac{y}{x}\right)-\frac{1}{2}\ln\left(1+\frac{y}{x}\right)=\ln x+k \quad .}$$

(We can also replace k with $\ln A$.)

Often we need to rearrange the equation first to get a homogeneous form, as in the following example.

Example 7.10:

Solve

$$xy\frac{dy}{dx}-y^2=3x^2 \quad ,$$

given $y=1$ when $x=1$.

Solution:

Rearrange as

$$\frac{dy}{dx}=\frac{3x^2+y^2}{xy} \quad .$$

This is therefore a homogeneous DE. We substitute $y=xv$,

$$v+x\frac{dv}{dx}=\frac{3x^2+x^2v^2}{x^2v}=\frac{3+v^2}{v}=\frac{3}{v}+v \quad .$$

We can now turn the crank,

$$\begin{aligned}
x\frac{dv}{dx} &= \frac{3}{v} \quad , \\
\int v\frac{dv}{dx}\,dx &= \int\frac{3}{x}\,dx \quad , \\
\frac{v^2}{2} &= 3\ln x+k \quad , \\
1/2y^2/x^2 &= 3\ln x+k \quad , \\
y^2 &= 2x^2(3\ln x+k) \quad ,
\end{aligned}$$

which is the general solution. Imposing the condition that for $x=1$, $y=1$, we obtain $1=2(0+k)$, and therefore $k=1/2$. The solution is thus

$$\boxed{y^2=2x^2(2\ln x+1/2).}$$

7.3 Bernoulli's Equation

Bernoulli's equation take the form

$$\frac{dy}{dx}+p(x)y=q(x)y^n\,.$$

In order to solve it, we convert it to linear type. Multiply both sides by $y^{-n}(1-n)$,

$$(1-n)y^{-n}\frac{dy}{dx}+p(x)(1-n)y^{-n+1}=q(x)(1-n)\,.$$

Now substitute $z=y^{1-n}$, using

$$\frac{dz}{dx}=\frac{dz}{dy}\frac{dy}{dx}=(1-n)y^{-n}\frac{dy}{dx}\,.$$

This leads to the equation

$$\frac{dz}{dx}+(1-n)p(x)z=(1-n)q(x)\,.$$

If we then define $\tilde{p}(x)=(1-n)p(x)$ and $\tilde{q}(x)=(1-n)q(x)$, we have an equation of linear type, which can be dealt with through an itegrating factor.

Example 7.11:

Solve $\dfrac{dy}{dx}+\dfrac{1}{x}y=xy^2$.

Solution: